Chapter 7. Minimizing curves & field of extremals

Calculus versus Calculus of Variations

Local minima: Necessary conditions

Calculus in $\mathbb{R}^2$	Calculus of variations
$\nabla f(x^*) = \underset{\sim}{0}$	$\dfrac{\partial f}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial \dot{x}}\right) = 0$
$x^* \in \mathbb{R}^2$	$x^* \in C^1$
CRITICAL POINT	EXTREMAL

Local minima: Sufficient conditions

Calculus	Calculus of variations
Let $y = x^* + \varepsilon\eta$,	Let $y = x^* + \varepsilon\eta$,
where $x^*,\ \eta,\ y \in \mathbb{R}^2$	where $x^*,\ \eta,\ y \in C^1$
$0 \leq f(y) - f(x^*)$	$0 \leq J[y] - J[x^*]$
$= \varepsilon V_1 + \varepsilon^2 V_2 + O(\varepsilon^3)$	$= \varepsilon V_1 + \varepsilon^2 V_2 + O(\varepsilon^3)$
$V_1 = \nabla f(x^*)\eta$	$V_1 = \displaystyle\int_{t_0}^{t_1} \eta \left\{ f_x - \frac{d}{dt} f_{\dot{x}} \right\} dt$
$V_2 = \eta^T H \eta / 2$	$V_2 = \dfrac{1}{2} \displaystyle\int_{t_0}^{t_1} \left(\eta^2 f_{xx} + 2\dot{\eta}\eta f_{x\dot{x}} \right.$
H the Hessian of f at x^*	$\left. + \dot{\eta}^2 f_{\dot{x}\dot{x}} \right) dt.$
$V_1 = 0,\ V_2 > 0$	$V_1 = 0,\ V_2 > 0$
sufficient for a minimum	not sufficient for a minimum

Example 1. $J[x] = \int_0^1 \frac{1}{\dot{x}} dt, \quad x(0) = 0, x(1) = 1.$

Solution. Extremal is $x = t = x^*$ & gives $J = 1$. Consider $y = t + \varepsilon\eta$, $\eta(0) = \eta(1) = 0$. Then

$$\begin{aligned}
\triangle J &= J[y] - J[t] \\
&= \int_0^1 \left((1 + \varepsilon\dot{\eta})^{-1} - 1\right) dt = \int_0^1 (-\varepsilon\dot{\eta} + \varepsilon^2\dot{\eta}^2 - \varepsilon^3\dot{\eta}^3 \cdots)dt \\
&= \varepsilon^2 \int_0^1 \dot{\eta}^2 dt + 0(\varepsilon^3).
\end{aligned}$$

Thus, $V_2 > 0$. But $y = \begin{cases} 3t, & 0 \le t \le \frac{1}{2} \\ -t + 2, & \frac{1}{2} \le t \le 1 \end{cases}$ satisfies

$$J[y] = \int_0^{\frac{1}{2}} \frac{dt}{3} + \int_{\frac{1}{2}}^1 (-1)dt = -\frac{1}{3} < J[x^*].$$

However, this is not a D_1 minimizing curve because the corner conditions are not satisfied. So, although $V_2 > 0$, $x = x^* = t$ is not a minimizing curve.

So we have to adopt a more sophisticated approach, using the concept of Hilbert Integral and a Field of Extremals.

Want $\triangle J = J[y] - J[x^*] > 0$ for *all* $y = y(t)$ satisfying the end conditions.

- # ALL $y = x^*(t) + \eta(t)$;
- # $y(t)$ perhaps in D_1.

The trick is to express $\triangle J$ in another form whose sign is easier to determine. The solution to E–L equation involves two arbitrary constants: two parameter family of curves. Imposing the end conditions gives the constants.
E.g. example above

$$0 - \frac{d}{dt}\left(\frac{-1}{\dot{x}^2}\right) = 0 \Rightarrow \dot{x} = \text{ constant}$$
$$\Rightarrow x = kt + l.$$

Values $k = 1$, $l = 0$ give the extremal thro' $(0,0)$ and $(1,1)$.
Consider the 1-parameter family of extremals $x = t + l$. As well as containing $x = x^* = t$, this family is a simple cover of the plane: one and only one curve passes through each point. Call this a FIELD OF EXTREMALS.
Since only one extremal of the family passes through each point, there is a unique value of slope

$$\dot{x} = p(t, x)$$

at each point (t, x). Associated with the field of extremals is a SLOPE FUNCTION $p(t, x)$ satisfying at each (t, x)

$$\frac{\partial f(t, x, p)}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial p}(t, x, p)\right) = 0.$$

Example 2. Find a field of extremals for $J[x] = \int_1^2 \dot{x}^2 t^3 dt$, $x(1) = 0$, $x(2) = 3$.

Solution. General solution of the E–L eqn is $x = k/t^2 + l$ and the extremal satisfying the end conditions is $x = 4 - 4/t^2$. A 1- par. family of extremals comparing this is $x = l - 4/t^2$. For $t > 0$ this gives a simple cover of the half plane and so is a field with slope function

$$p(t, x) = 8/t^3.$$

Now $f(t, x, p) = p^2 t^3$, $\partial f/\partial x = 0$ and $\partial f/\partial p = 2pt^3$. Hence

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial p}\right) = 0 - \frac{d}{dt}(2pt^3) = -\frac{d}{dt}(16) = 0.$$

Example 3. *Find a filed of extremals for*

$$J[x] = \int_0^1 \dot{x}^{2m}\, dx, \quad m > 1 \quad \textit{an integer}$$

with $x(0) = 1$ *and* $x(1) = 2$.

Solution. The Euler-Lagrange equation is

$$0 - \frac{d}{dt}(2m\dot{x}^{2m-1}) = 0.$$

This implies that $\dot{x} = \text{constant}$, so

$$x = x^*(t) = kt + l$$

Using $x(0) = 1$ and $x(2) = 3$, we have

$$l = 1 \quad k + l = 2 \Rightarrow k = 1.$$

Let $p = p(t,x) = \dot{x}^*(t) = 1$ be the slop of fields of extremals $x = t + l$. Then

$$f(t,x,p) = p^{2m} = 1, \quad \frac{\partial f}{\partial p} = 2mp^{2m-1} = 2m$$

which satisfies obviously the equation

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial p}\right) = 0$$

for all $(t,x) \in \mathbb{R}$.

Hilbert's Invariant Integral.

$\mathcal{C}^* : x = x^*(t)$ extremal in a field of extremals, slope $p(t, x)$.

$\mathcal{C} : x = x(t)$ any other curve joining the endpoints and covered by the field.

$$K[x] = \int_{t_0}^{t_1} \left\{ f(t, x, p) + (\dot{x} - p)\frac{\partial f}{\partial p}(t, x, p) \right\} dt.$$

When $\mathcal{C} = \mathcal{C}^*$, so $\dot{x} = p(t, x^*)$, $K[x^*] = J[x^*]$.

In $K[x]$, let

$$u(t, x) = f(t, x, p(t, x)) - p(t, x)\frac{\partial f}{\partial p}(t, x, p(t, x)),$$

$$v(t, x) = \frac{\partial f}{\partial p}(t, x, p(t, x)).$$

Then

$$\int_{t_0}^{t_1} \dot{x} v \, dt = \int_{\mathcal{C}} v \, dx$$

and we get a line integral:

$$K[x] = \int_{\mathcal{C}} u\, dt + v\, dx.$$

This integral is independent of $\mathcal{C}$.

To see this, we need to show $\partial u/\partial x - \partial v/\partial t = 0$ at each point (t, x). But, as x, t vary on $\mathcal{C}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[f(t, x, p(t, x)) - p(t, x)\frac{\partial f}{\partial p}(t, x, p(t, x))\right]$$

$$= \frac{\partial f}{\partial x}(t, x, p(t, x)) + \frac{\partial p}{\partial x}(t, x)\frac{\partial f}{\partial p}(t, x, p(t, x))$$

$$- \frac{\partial p}{\partial x}(t, x)\frac{\partial f}{\partial p}(t, x, p(t, x)) - p(t, x)\frac{\partial^2 f}{\partial x \partial p}(t, x, p(t, x))$$

$$- p(t, x)\frac{\partial p}{\partial x}(t, x)\frac{\partial^2 f}{\partial p^2}(t, x, p(t, x))$$

$$= \frac{\partial f}{\partial x}(t, x, p(t, x)) - p(t, x)\frac{\partial^2 f}{\partial x \partial p}(t, x, p(t, x))$$

$$- p(t, x)\frac{\partial p}{\partial x}(t, x)\frac{\partial^2 f}{\partial p^2}(t, x, p(t, x))$$

$$= \frac{\partial f}{\partial x}(t, x^*, p(t, x^*)) - p(t, x^*)\frac{\partial^2 f}{\partial x \partial p}(t, x, p(t, x^*))$$

$$- p(t, x^*)\frac{\partial p}{\partial x}(t, x^*)\frac{\partial^2 f}{\partial p^2}(t, x^*, p(t, x^*))$$

and

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial p}(t,x,p(t,x))\right]\\ &= \frac{\partial^2 f}{\partial t\partial p}(t,x,p(t,x)) + \frac{\partial^2 f}{\partial p^2}(t,x,p(t,x))\frac{\partial p}{\partial t}(t,x)\\ &= \frac{\partial^2 f}{\partial t\partial p}(t,x^*,p(t,x^*)) + \frac{\partial^2 f}{\partial p^2}(t,x^*,p(t,x^*))\frac{\partial p}{\partial t}(t,x^*),\end{aligned}$$

where we use the fact that $(t,x) = (t,x^*(t))$ for an extremal x^*. Using above two identities, we have

$$\begin{aligned}&\frac{\partial u}{\partial x} - \frac{\partial v}{\partial t}(t,x)\\ &= \frac{\partial f}{\partial x}(t,x^*,p(t,x^*)) - p(t,x^*)\frac{\partial^2 f}{\partial x\partial p}(t,x,p(t,x^*))\\ &\quad - p(t,x^*)\frac{\partial p}{\partial x}(t,x^*)\frac{\partial^2 f}{\partial p^2}(t,x^*,p(t,x^*))\\ &- \frac{\partial^2 f}{\partial t\partial p}(t,x^*,p(t,x^*)) - \frac{\partial^2 f}{\partial p^2}(t,x^*,p(t,x^*))\frac{\partial p}{\partial t}(t,x^*)\end{aligned} \tag{6.1}$$

Since $x = x^*(t)$ is an extremal in the field of extremals,

$$\frac{dx^*}{dt}(t) = p(t,x^*).$$

Using this equation, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial f}{\partial p}(t,x^*,p(t,x^*))\right] \\
&= \frac{\partial^2 f}{\partial p \partial t}(t,x^*,p(t,x^*)) + \frac{dx^*}{dt}\frac{\partial^2 f}{\partial p \partial x}(t,x^*,p(t,x^*)) \\
&\quad + \frac{d}{dt}\left[p(t,x^*(t))\right]\frac{\partial^2 f}{\partial p^2}(t,x^*,p(t,x^*)) \\
&= \frac{\partial^2 f}{\partial p \partial t}(t,x^*,p(t,x^*)) + \frac{dx^*}{dt}\frac{\partial^2 f}{\partial p \partial x}(t,x^*,p(t,x^*)) \\
&\quad + \left[\frac{\partial p}{\partial t}(t,x^*(t)) + \frac{dx^*}{dt}\frac{\partial p}{\partial x}(t,x^*(t))\right]\frac{\partial^2 f}{\partial p^2}(t,x^*,p(t,x^*)) \\
&= \frac{\partial^2 f}{\partial p \partial t}(t,x^*,p(t,x^*)) + p(t,x^*)\frac{\partial^2 f}{\partial p \partial x}(t,x^*,p(t,x^*)) \\
&\quad + \left[\frac{\partial p}{\partial t}(t,x^*(t)) + p(t,x^*)\frac{\partial p}{\partial x}(t,x^*(t))\right]\frac{\partial^2 f}{\partial p^2}(t,x^*,p(t,x^*))
\end{aligned} \tag{6.2}$$

From (6.1) and (6.2), we have

$$\begin{aligned}
&\frac{\partial u}{\partial x} - \frac{\partial v}{\partial t}(t,x) \\
&= \frac{\partial f}{\partial x}(t,x^*,p(t,x^*)) - \frac{d}{dt}\left[\frac{\partial f}{\partial p}(t,x^*,p(t,x^*))\right] = 0
\end{aligned}$$

since $p(t,x^*)$ is the slope of the field of extremals.

Since $K[x^*] = J[x^*]$, this gives

$$\begin{aligned} \triangle J = J[x] - J[x^*] &= J[x] - K[x^*] \\ &= J[x] - K[x]. \end{aligned}$$

Both integrals are evaluated along $\mathcal{C}$, so:

$$\triangle J = \int_{t_0}^{t_1} \left\{ f(t,x,\dot{x}) - f(t,x,p) - (\dot{x} - p)\frac{\partial f}{\partial p}(t,x,p) \right\} dt$$

- # $\dot{x}$ slope of $\mathcal{C}$ at (t,x);
- # p slope of *field of extremals* at (t,x).

Let

$$E(t,x,\dot{x},p) = f(t,x,\dot{x}) - f(t,x,p) - (\dot{x} - p)\frac{\partial f}{\partial p}(t,x,p)$$

denote the integrand in the integral defining $\triangle J$.
(Weierstrass Excess Function)

Theorem A (Weierstrass Conditions)
In order that the extremal $\mathcal{C}^* : x = x^*(t)$ give a strong local minimum to $J[x]$ it is sufficient that

- # $\mathcal{C}^*$ is a member of a field of extremals;
- # $E(t,x,\dot{x},p) \geq 0 \ \forall (t,x)$ close to $\mathcal{C}^*$ and arbitrary values of $\dot{x}$.

Theorem B (Weierstrass Conditions)
In order that the extremal $\mathcal{C}^* : x = x^*(t)$ give a (global) minimum to $J[x]$ it is sufficient that

- # $\mathcal{C}^*$ is a member of a field of extremals and the field of extremals cover the whole (t, x)-plan $\mathbb{R}^2$.
- # $E(t, x, \dot{x}, p) \geq 0 \; \forall (t, x) \in \mathbb{R}^2$.

Example 4. Shortest distance between 2 points is a straight line:

$$\text{minimize} \quad J = \int_0^1 (1 + \dot{x}^2)^{1/2} dt, \qquad \begin{array}{l} x(0) = 0 \\ x(1) = 1. \end{array}$$

Solution. Extremals are $x = kt + l$, end conditions $k = 1$, $l = 0 \Rightarrow x = t$. First, embed $x = t$ in a field of extremals. Could do $x = t + l$ (with $p(t, x) = 1$) or $x = kt$ ($p = x/t$). Now

$$\begin{aligned} E(t, x, \dot{x}, p) = f(t, x, \dot{x}) - f(t, x, p) \\ - (\dot{x} - p) \frac{\partial f}{\partial p}(t, x, p) \end{aligned}$$

$(p = p(t, x))$.

Here,

$$\begin{aligned} E(t, x, \dot{x}, p) &= (1 + \dot{x}^2)^{1/2} - (1 + p^2)^{1/2} \\ &\quad - p(\dot{x} - p)(1 + p^2)^{-1/2} \\ &= (1 + \dot{x}^2)^{1/2} - (1 + p\dot{x})(1 + p^2)^{-1/2}. \end{aligned}$$

At a point (t, x) covered by the field, $p = p(t, x)$ has a numerical value. We need to show that $E \geq 0$, for all possible values of $\dot{x}$, at such a point.

- If $1 + p\dot{x} \leq 0$, then $E > 0$.
- $1 + p\dot{x} > 0$; define

$$\begin{aligned} G &= (1 + \dot{x}^2)^{1/2} + (1 + p\dot{x})(1 + p^2)^{-1/2} > 0. \\ GE &= (1 + \dot{x}^2) - (1 + p\dot{x})^2/(1 + p^2) \\ &= \frac{\dot{x}^2 + p^2 - 2p\dot{x}}{1 + p^2} = \frac{(\dot{x} - p)^2}{1 + p^2} \geq 0. \end{aligned}$$

So $E \geq 0$ when $1 + p\dot{x} > 0$ and when $1 + p\dot{x} \leq 0$, and by the theorem $x = x^*(t) = t$, is a minimizing curve.

\# Field of extremals $x(t)$:

- Solutions of E–L eqn;
- Contain extremal $x^*(t)$ satisfying endpoints;
- Covers plane;
- Slope function

$$p(t,x) \text{ at each } (t,x).$$

\# WEIERSTRASS EXCESS FUNCTION $x = x(t)$

$$\begin{aligned} E(t,x,\dot{x}p) &= f(t,x,\dot{x}) - f(t,x,p) \\ &\quad - (\dot{x}-p)\frac{\partial f}{\partial p}(t,x,p) \end{aligned}$$

$$\begin{aligned} \dot{x} &= \text{slope of } x(t) \text{ at } (t,x) \\ p &= \text{slope function of field at } (t,x) \end{aligned}$$

\# $E(t,x,\dot{x},p) \geq 0 \quad \forall (t,x)$ close to $x = x^*(t)$ and all $\dot{x}$.

This

- # EXAMPLES
- # SEMIFIELDS & JACOBI CONDITIONS.

Example 2: (continue)

Minimizes $J[x] = \int_1^2 \dot{x}^2 t^3 dt$, $x(1) = 0$, $x(2) = 3$.

Solution. Extremal $x = k/t^2 + l$; satisfying end points, $x = 4 - 4/t^2$.

- # Field of extremals $x = l - 4/t^2$; slope of field $p(t,x) = 8/t^3$.
- # Excess function $f(t,x,\dot{x}) - f(t,x,p) - (\dot{x} - p)\frac{\partial f}{\partial p}(t,x,p)$

$$\begin{aligned} E(t,x,\dot{x},p) &= \dot{x}^2 t^3 - p^2 t^3 - (\dot{x} - p)2pt^3 \\ &= t^3(\dot{x}^2 - 2p\dot{x} + p^2) \\ &= t^3(\dot{x} - p)^2 \geq 0 \qquad t \geq 0 \end{aligned}$$

at each (t,x) for any value of $\dot{x}$. Hence the extremal satisfies the conditions and $x = x^*(t) = 4 - 4/t^2$ is a minimizing curve.