5.1

- x^* an extremum of $J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt$

  $\Leftrightarrow x^*$ satisfies Euler-Lagrange (E-L):

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

  If f independent of t then E-L $\Leftrightarrow$

$$\boxed{f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{ constant}}$$

- Solutions of $E-L$ (satisfying side conditions) called EXTREMALS. Seek minimizing curves among the extremals.

- **Examples**.

Ex 2. Minimize $\int \frac{\sqrt{1+\dot{x}^2}}{x^{1/2}}dt.$
Use the second form of the $E-L$ equ.

$$\left[\frac{(1+\dot{x}^2)}{x}\right]^{1/2} - \dot{x}\frac{\dot{x}}{[x(1+\dot{x}^2)]^{1/2}} = \text{ const.} \tag{5.1}$$

$$\{x(1+\dot{x}^2)\}^{-1/2}\{(1+\dot{x}^2)-\dot{x}^2\} = \text{ const.}$$

Hence

$$x(1+\dot{x}^2)^{1/2} = c. \tag{5.2}$$

Where $(1+\dot{x}^2)^{1/2}$ occurs, we use a parametric technique to solve the DE.
Substitute $\dot{x} = \tan\theta$.

Thus $(1+\dot{x}^2) = (1+\tan^2\theta) = \sec^2\theta = \frac{1}{\cos^2\theta}$.

From (5.2),

$$x = c\cos^2\theta. \qquad (5.3\text{”})$$

This gives x in terms of θ. What we will now do is obtain t in terms of θ.

Differentiate (5.3) and obtain

$$\dot{x} = \frac{d}{d\theta}\left(c\cos^2\theta\right)\frac{d\theta}{dt} = -2c\cos\theta\sin\theta\dot{\theta} \text{ but } \dot{x} = \tan\theta \text{ so}$$

$$\tan\theta = -2c\cos\theta\sin\theta\dot{\theta} \;\Leftrightarrow$$

$$1 = -2c\dot{\theta}\cos^2\theta = -c\dot{\theta}(1+\cos 2\theta)$$

$$= -c\dot{\theta}(1+\cos 2\theta), \text{ a first order separable equation for } \dot{\theta} \text{ so}$$

$$\int dt = -c\int(1+\cos 2\theta)d\theta + K$$

$$t = -c(\theta + \frac{1}{2}\sin 2\theta) + K.$$

Finally, setting $k = c/2$, $l = K$, and noting that $x = c\cos^2\theta = c(1+\cos 2\theta)/2$ we obtain

$$\boxed{\begin{aligned} x &= k(1+\cos 2\theta) \\ t &= l - k(2\theta + \sin 2\theta) \end{aligned}}$$

l, k to be found from end conditions.

Important features of this example:

$$\# \;\; f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{ const.} \Leftrightarrow \; E - L \text{ eqn;}$$

$\#$ $(1+\dot{x}^2)^{1/2}$ occurs $\Rightarrow$ substitute $\dot{x} = \tan\theta$
get a *parametric form* for $x^*(t)$.

Examples 3. Find the extremal of

$$J[x] = \int_0^1 (t\dot{x} + \dot{x}^2)dt, \quad x(0) = 1,$$
$$x(1) = 2.75$$

$E - L$ eqn:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left[\frac{\partial f}{\partial \dot{x}}\right] = 0$$

$$0 - \frac{d}{dt}(t + 2\dot{x}) = 0$$
$$\Rightarrow t + 2\dot{x} = 2c \qquad \text{(a constant)}$$
$$\Rightarrow \dot{x} = c - \frac{1}{2}t$$
$$\Rightarrow x = ct - \frac{1}{4}t^2 + k$$
$$x(0) = 1 \Rightarrow k = 1\,; \quad x(1) = 2.75 = c - .25 + 1$$
$$c = 2$$

Extremal

$$\boxed{x = 2t - \frac{1}{4}t^2 + 1.}$$

Example 4.

$$J[x] = \int_0^1 (1+t^2)\dot{x}^2 dt,$$
$$x(0) = \frac{\pi}{2}, \quad x(1) = \pi.$$

$E - L$ eqn:

$$0 - \frac{d}{dt}\{2(1+t^2)\dot{x}\} = 0$$
$$(1+t^2)\dot{x} = c$$
$$\int dx = \int \frac{c}{1+t^2} dt + D$$
$$x = c \arctan t + D$$
$$x(0) = \frac{\pi}{2} = D\,;\; x(1) = \pi = c \arctan 1 + \frac{\pi}{2}$$
$$= c.\frac{\pi}{4} + \frac{\pi}{2}$$
$$c = 2$$

Extremal $$\boxed{x(t) = 2\arctan t + \frac{\pi}{2}}$$

Ex.5.

$$J[x] = \int_0^{t_f} x\dot{x}^2 dt, \qquad x(0) = x_0$$
$$(x > 0) \qquad\qquad x(t_f) = x_f$$

Here t_f is fixed.

Solution. $f = x\dot{x}^2$ does not involve t explicitly.
So $E - L \Leftrightarrow$

$$f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{ const.}$$
$$x\dot{x}^2 - \dot{x}.2x\dot{x} = \text{ const.}$$
$$x\dot{x}^2 = c, \qquad \dot{x}^2 = \frac{c}{x}$$

Let

$$\alpha = \sqrt{c}\,, \quad \dot{x} = \alpha/\sqrt{x}$$
$$\int \sqrt{x}dx = \int \alpha dt + \beta$$
$$\frac{2}{3}x^{3/2} = \alpha t + \beta$$
$$t = 0\ : \qquad \frac{2}{3}x_0^{3/2} = \beta$$
$$t = t_f\ : \qquad \frac{2}{3}x_f^{2/3} = \alpha t_f + \frac{2}{3}x_0^{3/2}$$
$$\alpha = \frac{2}{3}\left[x_f^{3/2} - x_0^{3/2}\right]/t_f.$$

Fixed Endpoint Minimize $J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt$ where $(t_0, x(t_0))$ and $(t_1, x(t_1))$ are fixed. Let $x^*(t)$ be a minimizing curve. Let

$$y = x^*(t) + \varepsilon\eta(t)$$

be a weak variation starting at $(t_0, x(t_0))$ and ending at $(t_1, x(t_1))$.

We write $0(s)$ for a term T if $|T/s| = |O(s)/s| \leq k$ for some constant k and s small. In this case we say the term T is of order big O of s for small s. Thus $\lim_{s\to 0} O(s^2)/s = 0$.

We will show that the Euler-Lagrange Equation must be satisfied. Consider

$$0 \leq J[y] - J[x^*] = \triangle J$$

$$= \int_{t_0}^{t_1} f(t, x^* + \varepsilon\eta, \dot{x}^* + \varepsilon\dot{\eta})dt - \int_{t_0}^{t_1} f(t, x^*, \dot{x}^*)dt$$

$$= \int_{t_0}^{t_1} \left(f(t, x^* + \varepsilon\eta, \dot{x}^* + \varepsilon\dot{\eta}) - f(t, x^*, \dot{x}^*)\right) dt$$

$$= \int_{t_0}^{t_1} \left(f_x(t, x^*, \dot{x}^*)\epsilon\eta + f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\dot{\eta}\right) dt + 0(\epsilon^2),$$

by Taylor's Theorem below

$$= \int_{t_0}^{t_1} \left(f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*)\right) \epsilon\eta dt + [f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\eta]_{t_0}^{t_1}$$

$+ 0(\epsilon^2)$ integrating by parts...see below

$$= \int_{t_0}^{t_1} \left(f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*)\right) \epsilon\eta dt + 0(\epsilon^2)$$

since $\eta(t_0) = 0 = \eta(t_1)$

Since this holds for any function η satisfying $\eta(t_0) = 0 = \eta(t_1)$ and any ϵ it follows that Euler Equation hold:

$$\boxed{f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*) = 0}$$

NOTE: By Taylor's Theorem

$$f(t, x^* + \varepsilon\eta, \dot{x}^* + \varepsilon\dot{\eta}) - f(t, x^*, \dot{x}^*)$$
$$= f_x(t, x^*, \dot{x}^*)\epsilon\eta + f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\dot{\eta} + 0(\epsilon^2).$$

NOTE: Integration by parts

$$\int_{t_0}^{t_1} u\dot{v}dt = [uv]_{t_0}^{t_1} - \int_{t_0}^{t_1} v\dot{u}dt.$$

Setting $u = f_{\dot{x}}(t, x^*, \dot{x}^*)$ and $\dot{v} = \epsilon\dot{\eta}$, so that $v = \epsilon\eta$ and $\dot{u} = \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*)$, we obtain

$$\int_{t_0}^{t_1} f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\dot{\eta}dt = [f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\eta]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*)dt$$

Lemma 5.1. *If* $m(t) : [t_0, t_1] \to \mathbb{R}$ *is a continuous function and if*

$$\int_{t_0}^{t_1} m(t)\eta(t)\,dt = 0$$

for all C^2*-function* $\eta(t)$ *with* $\eta(t_0) = \eta(t_1) = 0$*. Then* $m(t) = 0$ *for all* $t \in [t_0, t_1]$*.*

Proof. Assume that there is some $a \in (t_0, t_1)$ so that $m(a) \neq 0$ (> 0). By the continuity of m, we can choose a sufficiently small $\varepsilon > 0$ such that $m(t) > 0$ (or $m(t) < 0$) for all $t \in [a - \varepsilon, a + \varepsilon]$. Choose

$$\eta(t) = \begin{cases} (t - a + \varepsilon)^3 (a + \varepsilon - t)^3 & \forall t \in [a - \varepsilon, a + \varepsilon] \\ 0 & \forall t \in [t_0, a - \varepsilon) \cup (a + \varepsilon, t_1] \end{cases}$$

Then

$$\int_{t_0}^{t_1} m(t)\eta(t)\,dt = \int_{a-\varepsilon}^{a+\varepsilon} m(t)\eta(t)\,dt > 0$$

This is impossible. $\square$

5.2 Variable Endpoint

Minimize $J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt$
$(t_0, x(t_0))$ is fixed, $(t_1, x(t_1))$ lies on given curve $x = c(t)$
(t_1 unknown)

Let $x^*(t)$ be a minimizing curve and suppose that it intersects the target curve at $t = t_1$. Let

$$y = x^*(t) + \varepsilon\eta(t)$$

be a weak variation starting at $(t_0, x(t_0))$ and reaching the curve at

$$t = t_1 + \triangle\tau.$$

We will show that the following transversality condition must be satisfied.

Transversal condition $$\boxed{f(t_1) + (\dot{c}(t_1) - \dot{x}^*(t_1))\frac{\partial f}{\partial \dot{x}}(t_1) = 0}$$

Solving $E-L$ eqn, gives two unknown constants. Condition $x(t_0) = x_0$ gives one equation, the transversality condition gives another equation **so** arbitrary constants can be found.

$$
\begin{aligned}
y(t_1 + \triangle\tau) &= x^*(t_1 + \triangle\tau) + \epsilon\eta(t_1 + \triangle\tau) \\
&= x^*(t_1) + \triangle\tau\dot{x}^*(t_1) + \epsilon\eta(t_1) + 0(\epsilon^2) \\
&= c(t_1 + \triangle\tau) = c(t_1) + \triangle\tau\dot{c}(t_1) + 0(\epsilon^2)
\end{aligned}
$$

Noting $x^*(t_1) = c(t_1)$, ignoring $O(\epsilon^2)$ terms and using the previous equation we obtain

$$
\epsilon\eta(t_1) = [\dot{c}(t_1) - \dot{x}^*(t_1)]\triangle\tau. \tag{1}
$$

Consider

$$
\begin{aligned}
\triangle J &= J[y] - J[x^*] \\
&= \int_{t_0}^{t_1+\triangle\tau} f(t, x^* + \epsilon\eta, \dot{x}^* + \epsilon\dot{\eta})dt - \int_{t_0}^{t_1} f(t, x^*, \dot{x}^*)dt \\
&= \int_{t_1}^{t_1+\triangle\tau} f(t, x^* + \epsilon\eta, \dot{x}^* + \epsilon\dot{\eta})dt \\
&+ \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x}(t, x^*, \dot{x}^*)\epsilon\eta + \frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*)\epsilon\dot{\eta} \right) dt + 0(\epsilon^2)
\end{aligned}
$$

Noting

$$\int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x}(t, x^*, \dot{x}^*)\epsilon\eta + \frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*)\epsilon\dot{\eta} \right) dt$$

$$= \int_{t_0}^{t_1} \frac{\partial f}{\partial x}(t, x^*, \dot{x}^*)\epsilon\eta dt + \left[\frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*)\epsilon\eta \right]_{t_0}^{t_1}$$

$$- \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*)\epsilon\eta dt$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial f}{\partial x}(t, x^*, \dot{x}^*) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*) \right] \epsilon\eta dt + [f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\eta]_{t_0}^{t_1}$$

and that

$$\triangle J = \int_{t_1}^{t_1+\triangle\tau} f(t, x^* + \epsilon\eta, \dot{x}^* + \epsilon\dot{\eta})dt$$

$$+ \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x}(t, x^*, \dot{x}^*)\epsilon\eta + \frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*)\epsilon\dot{\eta} \right) dt + 0(\epsilon^2)$$

$$= f(t_1, x^*, \dot{x}^*)\triangle\tau + \int_{t_0}^{t_1} \left[\frac{\partial f}{\partial x}(t, x^*, \dot{x}^*) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x^*, \dot{x}^*) \right] \epsilon\eta dt$$

$$+ [f_{\dot{x}}(t, x^*, \dot{x}^*)\epsilon\eta]_{t_0}^{t_1} + 0(\epsilon^2)$$

$$= f(t_1, x^*, \dot{x}^*)\triangle\tau + f_{\dot{x}}(t_1, x^*, \dot{x}^*)\epsilon\eta(t_1) + 0(\epsilon^2).$$

Here we used x^* is a solution of $E-L$ eqn and $\eta(t_0) = 0$.

Setting $f(t_1) := f(t_1, x^*(t_1), \dot{x}^*(t_1))$ and substituting for $\epsilon\eta(t_1)$ from (1) we obtain

$$\triangle J = \triangle\tau\{f(t_1) + [\dot{c}(t_1) - \dot{x}^*(t)]\frac{\partial f}{\partial \dot{x}}(t)\} + 0(\epsilon^2).$$

Note $\frac{d(\triangle J)}{d\varepsilon} = 0$ is called the first variation in J. Noticing

$$\triangle\tau = \frac{\epsilon\eta(t_1)}{[\dot{c}(t_1) - \dot{x}^*(t_1)]}$$

we show that if $x^*(t)$ is a minimizer, then $x^*(t)$ satisfies **transversality** and Euler-Lagrange equation.

Definition. *An extremal of minimizing $J[x]$ with variable endpoint* $\Leftrightarrow$ **transversality** + *$E - L$ equation.*

Example 1. Find the extremal of $\displaystyle\int_1^T \dot{x}^2 t^3 dt$, $x(1) = 0$, $T > 1$ is finite and $x(T)$ lies on $x = c(t) = \frac{2}{t^2} - 3$.

Solution. From Ex.1 of the previous chapter 4, extremals have the form

$$x = \frac{k}{t^2} + l.$$

$$x(1) = 0 \Rightarrow l = -k \Rightarrow x = \frac{k}{t^2} - k.$$

Use the transversality condition to find the 2 unknowns k, T. Now $\dot{x} = -\frac{2k}{t^3}$ and $x(t) = c(t) = \frac{2}{t^2} - 3$ when $t = T$, so $\dot{c}(t) = -\frac{4}{t^3}$.

Transversality Condition

$$f(t_1) + [\dot{c}(t_1) - \dot{x}^*(t_1)]\frac{\partial f}{\partial \dot{x}}(t_1) = 0$$

$$f(t) = \dot{x}^2 t^3\,, \quad \text{replace } t_1 \text{ by } T:$$

$$\dot{x}(T)^2 T^3 + 2\dot{x}(T)T^3[-4/T^3 + 2k/T^3] = 0$$

$$-\frac{2k}{T^3}T^3 + 2T^3[-4/T^3 + 2k/T^3] = 0$$

$$2k - 8 = 0\,, \quad k = 4.$$

Still have to determine T:

$$x(T) \text{ lies on } x = c(t) = \frac{2}{t^2} - 3$$

$$x(T) = 2/T^2 - 3 = \frac{4}{T^2} - 4$$

$$\frac{2}{T^2} = 1 \quad T = \sqrt{2} \qquad (T > 1, \text{ so not } -\sqrt{t}).$$

Consequently $x^*(t)$ meets target curve at $T = \sqrt{2}$, when $x(T) = -2$.

Note that here there were 3 unknowns: 2 arb. constants and T.
Three conditions

- $x(1) = 0$,
- Transversality condition,
- $x(T) = c(T)$.

Special Forms of Transversality:

If $x(t_1)$ is fixed and t_1 is variable.

Target curve here is $x = c(t) =$ const. $\dot{c}(t) = 0$ and the transversality cond– simplifies to

$$\boxed{f(t_1) - \dot{x}^*(t_1)\frac{\partial f}{\partial \dot{x}}(t_1) = 0}$$

$$\boxed{x(t_1) = x_1}$$

Able to solve for 2 constants and t_1

t_1 fixed and x_1 variable

Target curve is parallel to x-axis, $\dot{c}(t_1)$ is infinite.

If $\dot{c}(t_1)$ is infinite, trans. condn.

$$\frac{1}{\dot{c}(t_1)}f(t_1) + \left[1 - \frac{\dot{x}^*(t_1)}{\dot{c}(t_1)}\right]\frac{\partial f}{\partial \dot{x}}(t_1) = 0.$$

Let $\dot{c}(t_1) \to \infty$ here, transversality condn. simplifies to

$$\boxed{\frac{\partial f}{\partial \dot{x}}(t_1) = 0}$$

Example 2. Find the extremal of

$$J = \int_0^T (x^2 + \dot{x}^2)dt$$

when

$$\begin{array}{lll} (1) & x(0) = 1, & T = 2; \\ (2) & x(0) = 1, & x(T) = 2; \\ (3) & x(0) = 0, & x(T) = 2. \end{array}$$

Solution. Since f does not involve t explicitly, we could use the other form, but this is easy with the E-L eqn

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

$$2x - \frac{d}{dt}(2\dot{x}) = 0\,, \quad \ddot{x} - x = 0.$$

$$\lambda^2 - 1 = 0, \quad \lambda = -1, +1\,, \quad e^{-t}, e^t$$

$$x = Ae^t + Be^{-t}$$

(1) $x(0) = 1 \Rightarrow A + B = 1$. Transv. condn. is
$\dfrac{\partial f}{\partial \dot{x}}(T) = 0 \Rightarrow 2\dot{x}(T) = 0$ **i.e.** $\dot{x}(2) = 0$

$$Ae^2 - Be^{-2} = 0 \Rightarrow B = Ae^4$$

The extremal is $x = \frac{\cosh(t-2)}{\cosh 2}$,

$$x(2) = 1/\cosh 2.$$

(2) $x(0) = 1 \Rightarrow A + B = 1$. Trans. Condn.

$$f(T) - \dot{x}(T)\frac{\partial f}{\partial \dot{x}}(T) = 0$$
$$(Ae^T + Be^{-T})^2 + (Ae^T - Be^{-T})^2 - 2(Ae^T - Be^{-T})^2 = 0$$
$$2AB + 2AB = 0 \Rightarrow AB = 0.$$

So one of A, B must be zero

$$\Rightarrow x = e^t \quad \text{or} \quad x = e^{-t}.$$

But $x(T) = 2 \Rightarrow 2 = e^T$ or $2 = e^{-T}$. However, if $2 = e^{-T}$, $T < 0$, which is impossible. Hence $2 = e^T$, $T = \ln 2$.
The extremal is $x = e^t$, which has $x = 2$ at time $T = \ln 2$.