4.1: Revision of ODE
2nd order, constant coefficients

\[ a\ddot{x} + b\dot{x} + cx = f(t) \]

• Solve homogeneous equation

\[ a\ddot{x} + b\dot{x} + cx = 0 \]

to give \( x_h(t) \) (two arbitrary constants)

• Find a “particular solution” of nonhomog.equ.

\( x_p(t) \)

• Complete solution

\[ x(t) = x_h(t) + x_p(t) \]

# Homog. Eqn. \( a\ddot{x} + b\dot{x} + cx = 0 \).

Look for solutions \( x = e^{\lambda t} \)

\[(a\lambda^2 + b\lambda + c)e^{\lambda t} = 0 \]

Characteristic eqn. \( a\lambda^2 + b\lambda + c = 0 \)

• 2 real distinct solutions \( \lambda_1, \lambda_2 \)

\[ e^{\lambda_1 t}, \quad e^{\lambda_2 t} \]

• \( x_h(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \)

• double root \( \lambda : e^{\lambda t}, te^{\lambda t} \)

\[ x_h(t) = e^{\lambda t}(A + Bt) \]
• complex solutions (complex conjugates)

\[ \lambda = \alpha \pm i\beta \quad ; \quad e^{\alpha t} \cos \beta t, \quad e^{\alpha t} \sin \beta t \]

\[ x_h(t) = A e^{\alpha t} \cos \beta t + B e^{\alpha t} \sin \beta t \]

**Ex 1.** \( \ddot{x} + x = 0 \), \( \lambda^2 + 1 = 0 \), \( \lambda = \pm i \)

\[ x(t) = A \cos t + B \sin t. \]

**Ex 2.**

\[ \ddot{x} - 2\dot{x} - 3x = 0 \]
\[ \lambda^2 - 2\lambda - 3 = 0 \quad (\lambda - 3)(\lambda + 1) = 0 \]
\[ \lambda = 3, \quad x = e^{3t} \quad \lambda = -1, \quad x = e^{-t} \]

\[ x_h(t) = Ae^{-t} + Be^{3t}. \]
4.2

There are a number of techniques for finding particular solutions and we’ll only look at

METHOD OF UNDETERMINED COEFFICIENTS

Only works for

\[ f(t) = \begin{cases} 
\text{Polynomials in } t \\
e^{kt} \\
\cos \gamma t, \sin \gamma t 
\end{cases} \]

and combinations of these.

We would look for a particular solution of the form

\[ x_p(t) = \begin{cases} 
K_0 + K_1 t + \cdots + K_q t^q \\
L \cos \gamma t + M \sin \gamma t \\
H e^{kt}. 
\end{cases} \]

In particular, if

(*) \hspace{1cm} f(t) = t + t^2 + 5 \cos \gamma t + e^{2t}.

we look for a particular solution of the form

\[ x_p(t) = K_0 + K_1 t + K_2 t^2 + L \cos \gamma t + M \sin \gamma t + H e^{2t}. \]

The coefficients are determined by differentiating and substituting on the D.E.
Example 1: $\ddot{x} + x = e^{2t} + t^2$

Step 1.

Homog. eqn. $\ddot{x} + x = 0$, \quad $\lambda^2 + 1 = 0$

$\lambda = \pm i$

Solutions $\cos t$, $\sin t$

$$x_h = A \cos t + B \sin t.$$  

Step 2. Undetermined coefficients

$$x_p = K_0 + K_1 t + K_2 t^2 + H e^{2t}$$  

$$\dot{x}_p = K_1 + 2K_2 t + 2H e^{2t}$$  

$$\ddot{x}_p = 2K_2 + 4H e^{2t}$$

Sub in DE:

$$2K_2 + 4H e^{2t} + K_0 + K_1 t + K_2 t^2 + H e^{2t}$$

$$= e^{2t} + t^2$$

$$s2K_2 + K_0 = 0$$ \quad equating constants

$$K_1 = 0$$ \quad equating coefficients of $t$

$K_2 = 1$, \quad $2 + K_0 = 0$, \quad $K_0 = -2$ equating coefficients of $t^2$

$$5H = 1; \quad H = \frac{1}{5}.$$ \quad equating coefficients of $e^{2t}$

$$x_p(t) = -2 + t^2 + \frac{1}{5} e^{2t}.$$
Step 3.

\[ x(t) = x_h(t) + x_p(t) \]

\[ = A \cos t + B \sin t - 2 + t^2 + \frac{1}{5} e^{2t}. \]

**Remark 1.** Undetermined coefficients will not work for \( f(t) \) not of the specific type.

**Remark 2. Modification Rule.**

If \( f(t) \) is part of the general solution \( x_h(t) \), modify by multiplying by \( t \).

**Example 2:** \( \ddot{x} - 2\dot{x} - 3x = e^{3t} \).

1. \( \lambda^2 - 2\lambda - 3 = 0, \quad (\lambda - 3)(\lambda + 1) = 0, \quad \lambda = -1, 3; \)
   \( e^{-t}, e^{3t}; \quad x_h(t) = Ae^{-t} + Be^{3t}. \)

2. \( x_p(t) = Hte^{3t} \) (modified)
   \( \dot{x}_p = H(3t + 1)e^{3t}, \quad \ddot{x}_p = H(9t + 6)e^{3t} \)

Subst. on DE

\[ H(9t + 6)e^{3t} - 2H(3t + 1)e^{3t} - 3Hte^{3t} = e^{3t} \]

\[ 6H - 2H = 1, \quad H = \frac{1}{4}, \quad x_p(t) = \frac{1}{4}te^{3t} \]

\[ x(t) = Ae^{-t} + Be^{3t} + \frac{1}{4}te^{3t}. \]

Complete solution.
4.3. Optimization problems for functionals.

We use \((t, x)\) for the coordinates of a point in the plane and \(x = x(t)\) to represent the equation of a plane curve. Let \((t_0, x_0)\) and \((t_1, x_1)\) be two fixed points in the plane. A curve joining \((t_0, x_0)\) and \((t_1, x_1)\) is represented by \(x = x(t)\) for \(t_0 \leq t \leq t_1\) with \(x(t_0) = x_0\) and \(x(t_1) = x_1\).

The length of a curve \(x = x(t)\) joining \((t_0, x_0)\) and \((t_1, x_1)\) is

\[
L[x] = \int_{t_0}^{t_1} \sqrt{1 + (\dot{x})^2} \, dt; \quad \text{with} \quad \dot{x} = \frac{dx}{dt}.
\]

Different type of optimization problems involving integrals of a family of curves:

(1) **The classical isoperimetric problem:**
A curve (closed) of length \(l\).
What is the curve which
which surrounds the greatest area?

Maximising “area” which depends an curve \(x(t)\) (of length \(l\)) that encloses it.

(2) **Minimizing energy**

\[
\gamma : x(t). \quad \text{What is the curve for which the work (energy) is a minimum for moving } P \text{ from } A \text{ to } B?
\]
(3) **Brachistochrone problem:**

A particle is sliding down the curve under gravity. What is the shape of the curve so that the particle falls from $A$ to $B$ in minimum time?

Each of these problems is an optimization problem. However, the “variables” are functions $x(t)$ which define the curve.
A functional on a curve $x(t)$ joining the point

$$(t_0, x_0) \text{ to } (t_1, x_1)$$

is

$$J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x}) dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1$$

Here we suppose that $f$ is differentiable with respect to each variable as many times as we require.

First the fixed end point problem $(t_0, x_0), (t_1, x_1)$ fixed.

Minimise $J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x}) dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1$.

That is, find a curve $x^*(t)$ satisfying

$$J[x] \geq J[x^*]$$

for all $x = x(t)$ satisfying the boundary conditions. where $x^*$ is called a (global) minimizing curve

\# "close", variation and admissible curves.
\# Necessary condition for minimum

EULER-LAGRANGE EQUATION
To find a necessary condition, we require some additional assumptions and ideas.

The **class of admissible curves:**

$x(t)$ twice continuously differentiable, $x(t_0) = x_0$, $x(t_1) = x_1$.

Seek minimising curve $x^*(t)$ in this class.

The **problem:** Find among the admissible curves, one $x^*$ that gives a local minimum of $J[x]$. That is

$$J[x] \geq J[x^*]$$

for all admissible $x$ close to $x^*$.

What do we mean by “close to” for curves $x(t)$?

**WEAK VARIATION.** Let $x^*(t)$ be a curve and $x(t)$ an admissible curve. If $\exists$ small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$|x^*(t) - x(t)| < \varepsilon_1 \quad \text{for all}$$

$$|\dot{x}^*(t) - \dot{x}(t)| < \varepsilon_2, \quad t_0 \leq t \leq t,$$

then $x(t)$ is said to be a weak variation of $x^*(t)$.

**Weak local minimizer.** A curve $x^* = x^*(t) : [t_0, t_1] \rightarrow \mathbb{R}$ is called “Weakly local minimizer of functional if there are small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$J(x^*) \leq J(y)$$

for all curve $y = y(t)$ satisfying the boundary conditions and

$$|x^*(t) - y(t)| < \varepsilon_1 \quad \text{for all}$$

$$|\dot{x}^*(t) - \dot{y}(t)| < \varepsilon_2, \quad t_0 \leq t \leq t,$$
**STRONG VARIATION.** $x^*(t), x(t)$ as above. There exists a small $\varepsilon > 0$ such that $|x^*(t) - x(t)| < \varepsilon$ for all $t_0 \leq t \leq t_1$. Then $x(t)$ is called a strong variation of $x^*(t)$.

**Strong local minimizer (minimizing curve).**

A curve $x^* = x^*(t) : [t_0, t_1] \to \mathbb{R}$ is called “Weakly local minimizer of functional $J$ if there is a small $\varepsilon$ such that

$$J[x^*] \leq J[y]$$

for all curve $y = y(t)$ satisfying the boundary conditions and

$$|x^*(t) - y(t)| < \varepsilon, \quad t_0 \leq t \leq t.$$

**Remark:** A global minimizing curve is a strong locally minimizing curve. A strong local minimizing curve is a weak local minimizing curve.

A first necessary condition for a “weak local minimum” (i.e. a local minimum with respect to weak variations).

**Theorem 4.1.** In order that $x = x^*(t)$ should be a minimizing curve, (i.e. Minimize $J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt$, $x(t_0) = x_0, x(t_1) = x_1$) in the class of $C^2$ functions to the fixed endpoint problem it is necessary that

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) = 0$$

at each point of $x = x^*(t)$. 
A few remarks about Theorem 4.1.

Let \( \eta : [t_0, t_1] \to \mathbb{R} \) be a \( C^1 \)-function of continuous first derivative with \( \eta(t_0) = \eta(t_1) = 0 \). Then consider

\[
F(\varepsilon) = J(x + \varepsilon \eta) = \int_{t_0}^{t_1} f(t, x + \varepsilon \eta, \dot{x} + \varepsilon \dot{\eta}) \, dt
\]

By Theorem 1.1, \( F'(0) = 0 \) since \( \varepsilon = 0 \) is a local minimal point of \( F \). It follows from the Chain rule that

(i) \[
F'(0) = \int_{t_0}^{t_1} \frac{\partial f}{\partial x} \eta + \frac{\partial f}{\partial \dot{x}} \frac{d\eta}{dt} \, dt = 0
\]

Integration by parts yields

(ii) \[
\int_{t_0}^{t_1} \frac{\partial f}{\partial \dot{x}} \eta' \, dt = \int_{t_0}^{t_1} \left( \frac{\partial f}{\partial \dot{x}} \right)' \eta \, dt - \int_{t_0}^{t_1} \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \eta \, dt
= - \int_{t_0}^{t_1} \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \eta \, dt
\]

due to the fact that \( \eta(t_0) = \eta(t_1) = 0 \).

From (i)-(ii), we have

(iii) \[
\int_{t_0}^{t_1} \left[ \frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) \right] \eta \, dt = 0
\]

for all \( \eta : [t_0, t_1] \to \mathbb{R} \) of \( C^1 \)-continuous functions with \( \eta(t_0) = \eta(t_1) = 0 \). \( \square \)
Equation (4.1) is called Euler-Lagrange equation. This is an ODE in $x$, since $f$ is known. Solutions of equation (4.1) are called *extremals*.

**EXAMPLES 1.** Find the extremal of

$$J[x] = \int_1^2 \dot{x}^2 t^3 dt, \quad x(1) = 0, \quad x(2) = 3$$

Solution

$$f(t, x, \dot{x}) = \dot{x}^2 t^3$$

Euler - Lagrange equation is \( \frac{df}{dt} \frac{\partial f}{\partial p} \) Here \( p = \dot{x} \), so \( f = f(t, x, p) \).

$$0 - \frac{d}{dt} \{2\dot{x}t^3\} = 0$$

So \( \dot{x}t^3 = \text{constant} = k \)

Separable equation:

$$\int dx = \int \frac{k}{t^3} dt + l$$

$$x = \frac{K}{t^2} + l$$

\( x(1) = 0 \Rightarrow K + l = 0 \)

\( x(2) = 3 \Rightarrow 3 = \frac{K}{4} - K \)

$$12 = -3K, \quad K = -4$$

$$x = 4 - \frac{4}{t^2}.$$
Solution is a curve joining 
(1, 0) and (2,3) which 
is a candidate for minimizing $J[x]$.

Example 2. Brachistochrone: 
Find the path of least 
time on which a particle 
descents from rest at $A$ 
to $B$ by gravity.

Solution. Take $A = (0, 0)$ as origin and measure vertically 
downwards and $B = (a, b)$. Say velocity $v$ at depth $x$ and 
arc-length

\[ v = \frac{ds}{dT} = \sqrt{1 + \dot{x}^2}, \quad s(t) = \int_0^t \sqrt{1 + \dot{x}^2(s)} \, ds \]

Energy is conserved

\[ v = \frac{ds}{dT} = \sqrt{2gx}. \]

Time from $A$ to $B$ is

\[ T[x] = \int_A^B \, dT = \int_A^B \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1 + \dot{x}^2}}{\sqrt{x}} \, dt. \]

Want to minimize

\[ J[x] = \int_0^a \frac{\sqrt{1 + \dot{x}^2}}{\sqrt{x}} \, dt. \]
Integrand does not depend explicitly on \( t \), then

\[
f - \dot{x} \frac{\partial f}{\partial x} = \text{const.}
\]

For, if \( f = f(x, \dot{x}) \),

\[
\frac{d}{dt} \left( f - \dot{x} \frac{\partial f}{\partial x} \right) = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial x} \ddot{x} - \dot{x} \frac{\partial f}{\partial \dot{x}} - \dot{x} \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) \\
= \dot{x} \left[ \frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) \right] = 0
\]

by Euler-Lagrange