4.1: Revision of ODE

2nd order, constant coefficients

$$a\ddot{x} + b\dot{x} + cx = f(t)$$

- Solve homogeneous equation

$$a\ddot{x} + b\dot{x} + cx = 0$$

  to give $x_h(t)$ (two arbitrary constants)
- Find a "particular solution" of nonhomog.equ. $x_p(t)$
- Complete solution

$$x(t) = x_h(t) + x_p(t).$$

\# Homog. Eqn. $a\ddot{x} + b\dot{x} + cx = 0$.
Look for solutions $x = e^{\lambda t}$

$$(a\lambda^2 + b\lambda + c)e^{\lambda t} = 0$$

Characteristic eqn. $a\lambda^2 + b\lambda + c = 0$

- 2 real distinct solutions λ_1, λ_2

$$e^{\lambda_1 t} \quad , \quad e^{\lambda_2 t}$$

- $x_h(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
- double root λ : $e^{\lambda t}$, $te^{\lambda t}$

$$x_h(t) = e^{\lambda t}(A + Bt)$$

- complex solutions (complex conjugates)

$$\lambda = \alpha \pm i\beta \quad ; \quad e^{\alpha t}\cos\beta t\,, \quad e^{\alpha t}\sin\beta t$$
$$x_h(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$$

Ex 1. $\ddot{x} + x = 0, \qquad \lambda^2 + 1 = 0, \qquad \lambda = \pm i$

$$x(t) = A\cos t + B\sin t.$$

Ex 2.

$$\ddot{x} - 2\dot{x} - 3x = 0$$
$$\lambda^2 - 2\lambda - 3 = 0 \qquad (\lambda - 3)(\lambda + 1) = 0$$
$$\lambda = 3, \quad x = e^{3t} \qquad \lambda = -1, \quad x = e^{-t}$$
$$x_h(t) = Ae^{-t} + Be^{3t}.$$

4.2

There are a number of techniques for finding particular solutions and we'll only look at

METHOD OF UNDETERMINED COEFFICIENTS

Only works for

$$f(t) = \begin{cases} \text{Polynomials in } t \\ e^{kt} \\ \cos\gamma t, \; \sin\gamma t \end{cases}$$

and combinations of these.

We would look for a particular solution of the form

$$x_p(t) = \begin{cases} K_0 + K_1 t + \cdots + K_q t^q \\ L\cos\gamma t + M\sin\gamma t \\ He^{kt}. \end{cases}$$

In particular, if

$$(*) \qquad f(t) = t + t^2 + 5\cos\gamma t + e^{2t}.$$

we look for a particular solution of the form

$$x_p(t) = K_0 + K_1 t + K_2 t^2 + L\cos\gamma t + M\sin\gamma t + He^{2t}.$$

The coefficients are determined by differentiating and substituting on the D.E.

Example 1: $\ddot{x} + x = e^{2t} + t^2$

Step 1.

$$\text{Homog. eqn. } \ddot{x} + x = 0, \qquad \lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

Solutions $\cos t$, $\sin t$

$$x_h = A\cos t + B\sin t.$$

Step 2. Undetermined coefficients

$$x_p = K_0 + K_1 t + K_2 t^2 + He^{2t}$$
$$\dot{x}_p = K_1 + 2K_2 t + 2He^{2t}$$
$$\ddot{x}_p = 2K_2 + 4He^{2t}$$

Sub in DE:

$$2K_2 + 4He^{2t} + K_0 + K_1 t + K_2 t^2 + He^{2t}$$
$$= e^{2t} + t^2$$
$$s2K_2 + K_0 = 0 \qquad \text{equating constants}$$
$$K_1 = 0 \qquad \text{equating coefficients of } t$$
$$K_2 = 1, \quad 2 + K_0 = 0, \qquad K_0 = -2 \text{ equating coefficients of } t^2$$
$$5H = 1\,; \quad H = \frac{1}{5}. \quad \text{equating coefficients of } e^{2t}$$
$$x_p(t) = -2 + t^2 + \frac{1}{5}e^{2t}.$$

Step 3.

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= A\cos t + B\sin t - 2 + t^2 + \frac{1}{5}e^{2t}. \end{aligned}$$

Remark 1. Undetermined coefficients will not work for $f(t)$ not of the specific type.

Remark 2. Modification Rule.

If $f(t)$ is part of the general solution $x_h(t)$, *modify* by multiplying by t.

Example 2: $\ddot{x} - 2\dot{x} - 3x = e^{3t}$.

1. $\lambda^2 - 2\lambda - 3 = 0$, $(\lambda - 3)(\lambda + 1) = 0$, $\lambda = -1, 3$; e^{-t}, e^{3t}; $x_h(t) = Ae^{-t} + Be^{3t}$.
2. $x_p(t) = Hte^{3t}$ (*modified*) $\dot{x}_p = H(3t+1)e^{3t}$, $\ddot{x}_p = H(9t+6)e^{3t}$

Subst. on DE

$$H(9t+6)e^{3t} - 2H(3t+1)e^{3t} - 3Hte^{3t} = e^{3t}$$

$$6H - 2H = 1\,, \quad H = \frac{1}{4}\,, \quad x_p(t) = \frac{1}{4}te^{3t}$$

$$x(t) = Ae^{-t} + Be^{3t} + \frac{1}{4}te^{3t}.$$

Complete solution.

4.3. Optimization problems for functionals.

We use (t, x) for the coordinates of a point in the plane and $x = x(t)$ to represent the equation of a plan curve. Let (t_0, x_0) and (t_1, x_1) be two fixed points in the plane. A curve joining (t_0, x_0) and (t_1, x_1) is represented by $x = x(t)$ for $t_0 \leq t \leq t_1$ with $x(t_0) = x_0$ and $x(t_1) = x_1$.

The length of a curve $x = x(t)$ joining (t_0, x_0) and (t_1, x_1) is

$$L[x] = \int_{t_0}^{t_1} \sqrt{1 + (\dot{x})^2}\, dt; \quad \text{with} \quad \dot{x} = \frac{dx}{dt}.$$

Different type of optimization problems involving integrals of a family of curves :

(1) **The classical isoperimetric problem:**
A curve (closed) of length l.
What is the curve which
which surrounds the greatest area?

Maximising "area" which depends an curve $x(t)$ (of length l) that encloses it.

(2) **Minimizing energy**

$\gamma : x(t)$. What is the curve for which the work (energy) is a mimimum for moving P from A to B?

(3) **Brachistochrone problem:**

A particle is sliding down the curve under gravity What is the shape of the curve so that the particle falls from A to B in minimum time?

Each of these problems is an optimization problem. However, the "variables" are *functions* $x(t)$ which define the curve.

A *functional* on a curve $x(t)$ joining the point

$(t_0,\, x_0)$ to $(t_1,\, x_1)$ is

$$J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt\,, \quad x(t_0) = x_0, \quad x(t_1) = x_1$$

Here we suppose that f is differentiable with respect to each variable as many times as we require.

First the **fixed end point problem** $(t_0,\, x_0)$, $(t_1,\, x_1)$ fixed.

Minimise $J[x] = \displaystyle\int_{t_0}^{t_1} f(t, x, \dot{x})dt$, $x(t_0) = x_0$, $x(t_1) = x_1$. That is, find a curve $x^*(t)$ satisfying

$$J[x] \geq J[x^*]$$

for all $x = x(t)$ satisfying the boundary conditions. where x^* is called a (global) minimizing curve

\# “close”, variation and admissible curves.

\# Necessary condition for minimum
EULER-LAGRANGE EQUATION

To find a necessary condition, we require some additional assumptions and ideas.

The **class of admissible curves:**
$x(t)$ twice continuously diff'able, $x(t_0) = x_0$, $x(t_1) = x_1$.
Seek minimising curve $x^*(t)$ in this class.

The **problem:** Find among the admissible curves, one x^* that gives a local minimum of $J[x]$. That is

$$J[x] \geq J[x^*]$$

for all admissible x *close to* x^*.
What do we mean by "close to" for curves $x(t)$?

WEAK VARIATION. Let $x^*(t)$ be a curve and $x(t)$ an admissible curve. If $\exists$ small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\begin{aligned} |x^*(t) - x(t)| &< \varepsilon_1 && \text{for all} \\ |\dot{x}^*(t) - \dot{x}(t)| &< \varepsilon_2, && t_0 \leq t \leq t, \end{aligned}$$

then $x(t)$ is said to be a weak variation of $x^*(t)$.

Weak local minimizer. *A curve $x^* = x^*(t) : [t_0, t_1] \to \mathbb{R}$ is called "Weakly local minimizer of functional if there are small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that*

$$J(x^*) \leq J(y)$$

for all curve $y = y(t)$ satisfying the boundary conditions and

$$\begin{aligned} |x^*(t) - y(t)| &< \varepsilon_1 && \textit{for all} \\ |\dot{x}^*(t) - \dot{y}(t)| &< \varepsilon_2, && t_0 \leq t \leq t, \end{aligned}$$

STRONG VARIATION. $x^*(t)$, $x(t)$ as above. There exists a small $\varepsilon > 0$ such that $|x^*(t) - x(t)| < \varepsilon$ for all $t_0 \leq t \leq t_1$. Then $x(t)$ is called a strong variation of $x^*(t)$.

Strong local minimizer (minimizing curve).

A curve $x^* = x^*(t) : [t_0, t_1] \to \mathbb{R}$ is called "Weakly local minimizer of functional J if there is a small ε such that

$$J[x^*] \leq J[y]$$

for all curve $y = y(t)$ satisfying the boundary conditions and

$$|x^*(t) - y(t)| < \varepsilon, \qquad t_0 \leq t \leq t.$$

Remark: A global minimizing curve is a strong locally minimizing curve. A strong local minimizing curve is a weak local minimizing curve.

A first necessary condition for a "weak local minimum" (i.e. a local minimum with respect to *weak variations*).

Theorem 4.1. *In order that $x = x^*(t)$ should be a minimizing curve, (i.e. Minimize $J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt$, $x(t_0) = x_0$, $x(t_1) = x_1$) in the class of C^2 functions to the fixed endpoint problem it is necessary that*

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \tag{4.1}$$

at each point of $x = x^*(t)$.

A few remarks about Theorem 4.1.

Let $\eta : [t_0, t_1] \to \mathbb{R}$ be a C^1-function of continuous first derivative with $\eta(t_0) = \eta(t_1) = 0$. Then consider

$$F(\varepsilon) = J(x + \varepsilon\eta) = \int_{t_0}^{t_1} f(t, x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta})\, dt$$

By Theorem 1.1, $F'(0) = 0$ since $\varepsilon = 0$ is a local minimal point of F. It follows from the Chain rule that

(i) $$F'(0) = \int_{t_0}^{t_1} \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial \dot{x}}\frac{d\eta}{dt}\, dt = 0$$

Integration by parts yields

(ii)
$$\begin{aligned}\int_{t_0}^{t_1} \frac{\partial f}{\partial \dot{x}}\eta'\, dt &= \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial \dot{x}}\eta\right)'\, dt - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\eta\, dt \\ &= -\int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\eta\, dt\end{aligned}$$

due to the fact that $\eta(t_0) = \eta(t_1) = 0$.

From (i)-(ii), we have

(iii) $$\int_{t_0}^{t_1} \left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\eta\, dt = 0$$

for all $\eta : [t_0, t_1] \to \mathbb{R}$ of C^1-continuous functions with $\eta(t_0) = \eta(t_1) = 0$. $\square$

Equation (4.1) is called Euler-Lagrange equation. This is an ODE in x, since f is known. Solutions of equation (4.1) are called *EXTREMALS*.

EXAMPLES 1. Find the extremal of

$$J[x] = \int_1^2 \dot{x}^2 t^3 dt, \quad \begin{array}{l} x(1) = 0 \\ x(2) = 3 \end{array}$$

Solution

$$f(t, x, \dot{x}) = \dot{x}^2 t^3$$

Euler - Lagrange equation is $\dfrac{d}{dt}\dfrac{\partial f}{\partial p}$ Here $p = \dot{x}$, so $f = f(t, x, p)$.

$$0 - \frac{d}{dt}\{2\dot{x}t^3\} = 0$$

So $\dot{x}t^3 = \text{ constant } = k$
Separable equation:

$$\int dx = \int \frac{k}{t^3} dt + l$$

$$x = \frac{K}{t^2} + l$$

$$\begin{aligned} x(1) = 0 \Rightarrow & K + l = 0 \\ x(2) = 3 \Rightarrow & 3 = \frac{K}{4} - K \\ & 12 = -3K, \quad K = -4 \\ x = & 4 - \frac{4}{t^2}. \end{aligned}$$

Solution is a curve joining
$(1,0)$ and (2,3) which
is a candidate for minimizing $J[x]$.

Example 2. **Brachistochrane**:
Find the path of least
time on which a particle
descents from rest at A
to B by gravity.

Solution. Take $A=(0,0)$ as origin and measure vertically downwards and $B=(a,b)$. Say velocity v at depth x and arc-length

$$v=\frac{ds}{dT}=\sqrt{1+\dot{x}^2},\quad s(t)=\int_0^t\sqrt{1+\dot{x}^2(s)}\,ds$$

Energy is conserved

$$v=\frac{ds}{dT}=\sqrt{2gx}.$$

Time from A to B is

$$T[x]=\int_A^B dT=\int_A^B\frac{ds}{v}=\frac{1}{\sqrt{2g}}\int_0^a\frac{\sqrt{1+\dot{x}^2}}{\sqrt{x}}dt.$$

Want to minimize

$$J[x]=\int_0^a\frac{\sqrt{1+\dot{x}^2}}{\sqrt{x}}dt.$$

\#   Integrand does not defend explicitly on t, Then

$$\boxed{f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{ const.}}$$

For, if $f = f(x, \dot{x})$,

$$\begin{aligned}\frac{d}{dt}\left(f - \dot{x}\frac{\partial f}{\partial \dot{x}}\right) &= \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial \dot{x}}\ddot{x} - \ddot{x}\frac{\partial f}{\partial \dot{x}} - \dot{x}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) \\ &= \dot{x}\left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right] = 0\end{aligned}$$

by Euler-Lagrange