3. Minimization with constraints

Problem III. *Minimize $f(x)$ in $\mathbb{R}^n$ given that x satisfies the equality constraints*

$$g_j(x) = c_j, \quad j = 1, ..., m < n,$$

where c_1, ..., c_m are given numbers.

Theorem 3.1. *Let $f(x)$ and $g_j(x)$ be defined and have continuous second derivatives in some open region of $\mathbb{R}^n$. Then necessary condition that a minimize $f(x)$ with the constraints*

$$g_j(x) = c_j, \quad j = 1, ..., m < n$$

is that there exist m-Lagrange multipliers λ_1,...,λ_m such that

$$grad\left(f + \sum_{j=1}^{m} \lambda_j g_j\right) = 0 \quad at\ a$$

(For the proof of Theorem 3.1, we refer to the book of E. R. Pinch.)

Example 3.1. *Minimize $f(x) = 1 - x_1^2 - x_2^2$ subject to $g(x) = x_2 - 1 + x_1^2 = 0$.*

Solution. Using Theorem 3.1 with $m = 1$ and $n = 2$, there is a Lagrange multiplier λ such that

$$\text{grad}(f + \lambda g) = 0$$

This is equivalent to

$$-2x_1 + 2\lambda x_1 = 0 \quad \text{and} \quad -2x_2 + \lambda = 0.$$

There are three unknowns so we need another equation (i.e the constrain itself):

$$x_2 - 1 + x_1^2 = 0.$$

Solving these equations we find the following solutions

$$x_1 = 0, \quad x_2 = 1, \quad \lambda = 2$$

and

$$x_1 = \pm\frac{1}{\sqrt{2}}, \quad x_2 = \frac{1}{2}, \quad \lambda = 1.$$

Sketch the constraint curve and level set of f. Then you find that the points $(0, 1)$ and $(\pm\frac{1}{\sqrt{2}}, \frac{1}{2})$ are the points where curves of level sets of f touch the parabola of constraint. It is clear that the minimum is at $x_1 = 0$ and $x_2 = 1$. (geometry behind?) $\square$

Example 3.2: Find local extremal of

$$f(x) = x_1^3 + x_2^3 + x_3^3$$

where

$$g_1(x) = x_1^2 + x_2^2 + x_3^2 - 5 = 0 \tag{1}$$

$$g_2(x) = x_1^2 + x_2^2 + x_3^2 - 2x_1 - 3 = 0 \tag{2}$$

- Lagrangian

$$
\begin{aligned}
L &= f + \lambda_1 g_1 + \lambda_2 g_2 \\
&= x_1^3 + x_2^3 + x_3^3 \\
&\quad + \lambda_1(x_1^2 + x_2^2 + x_3^2 - 5) \\
&\quad + \lambda_2(x_1^2 + x_2^2 + x_3^3 - 2x_1 - 3)
\end{aligned}
$$

- grad $L = 0$

$$3x_1^2 + 2\lambda_1 x_1 + 2\lambda_2(x_1 - 1) = 0 \tag{3}$$

$$3x_2^2 + 2\lambda_1 x_2 + 2\lambda_2 x_2 = 0 \tag{4}$$

$$3x_3^3 + 2\lambda_1 x_3 + 2\lambda_2 x_3 = 0 \tag{5}$$

- From (4), $x_2 = 0$ or $3x_2 + 2\lambda_1 + 2\lambda_2 = 0$
  From (5), $x_3 = 0$ or $3x_3 + 2\lambda_1 + 2\lambda_2 = 0$

If $x_2 = 0$

$$x_1^2 + x_3^2 - 5 = 0 \tag{1a}$$

$$x_1^2 + x_3^2 - 2x_1 - 3 = 0 \tag{2a}$$

$$-2x_1 + 2 = 0 \qquad (1(a) - 2(a))$$

$x_1 = 1$. Subst. this in (3) to obtain

$$3 + 2\lambda_1 + 2\lambda_2(0) = 0, \quad \lambda_1 = -3/2.$$

Subst. $x_1 = 1$ in $1(a)$, $x_3^2 = 4$, $x_3 = \pm 2$
$x_3 = +2$, in (5): $3.4 + (-3).2 + 2\lambda_2.2 = 0$

$$\Rightarrow \lambda_2 = -3/2$$

$x_3 = -2$, in (5) $3.4 + (-3)(-2) - 4\lambda_2 = 0$

$$\lambda_2 = 9/2$$

$$\left.\begin{array}{lll} (1,0,2), & \lambda_1 = -3/2, & \lambda_2 = -3/2 \\ (1,0,-2), & \lambda_1 = -3/2, & \lambda_2 = 9/2 \end{array}\right\} x_2 = 0.$$

If $x_3 = 0$: from the constraint equations

$$x_1^2 + x_2^2 - 5 = 0 \tag{1b}$$

$$x_1^2 + x_2^2 - 2x_1 - 3 = 0 \tag{2b}$$

Again $x_1 = 1$ and $\lambda_1 = -3/2$.
As before, substitute $x_1 = 1$ in (1b) $\Rightarrow x_2^2 = 4$, $x_2 = \pm 2$.
Obtain

$$\begin{array}{lll} (1,2,0), & \lambda_1 = -3/2, & \lambda_2 = -3/2 \\ (1,-2,0), & \lambda_1 = -3/2, & \lambda_2 = 9/2. \end{array}$$

These and the other solutions ($x_2 = x_3 \neq 0$) above are the critical points of the problem.
Are these maxima or minima?
We need *sufficient* conditions to say.

Distinguish minima from maxima (sufficient conditions)

$$\text{Minimise } f(x_1, x_2, ..., x_a) \quad \text{subject to}$$

$$g_1(x_1, ..., x_n) = c,$$

$$\vdots$$

$$g_m(x_1, ..., x_n) = c_m$$

$$L = f + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_m g_m.$$

Suppose that $x = a$ is a critical point

$$\# \qquad \text{grad } L(a) = 0.$$

Let H_L be the Hessian of L. This means that H_L involves $\lambda_1, ..., \lambda_m$ as well as the $a_1, ..., a_n$.

$$\# \qquad h^T H_L h \geq 0 \qquad \text{at } a$$

for all $h \neq 0$ such that h^T grad $g_i = 0$, for all $1 \leq i \leq m$.

For $g = (g_1, \cdots, g_m)$, we define

$$B = \nabla g = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \cdot & \cdots & \cdot \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

Bordered Hessian

$$H = \begin{pmatrix} H_L & B \\ B & O \end{pmatrix}, \text{ at } x = a.$$

is a $(m+n) \times (m+n)$-matrix, where O is a $m \times m$ zero matrix. Point a at which grad $L = 0$ and $\det H \neq 0$ is called a nondegenerate critical point.

Theorem 3.2. *: (Nec. & Suff. for a minimum)*

Let a be a non-degenerate critical point for f, subject to $g_i = c_i$, $i = 1, ..., m$. A necessary and sufficient condition that $x = a$ is a point where f has a local minimum subject to the constraints is that

$$h^T H_L h \geq 0$$

for all tangent vectors h. *
Sufficient condition for a local maximum is that

$$h^T H_L h \leq 0,$$

for all tangent vectors h. *
* *Recall vector h is tangent if h^T grad $g_i = 0$, for all i.*

To solve constrained problems

Construct L
Find critical points a where grad $L = 0$.
For $x = a$, check non-degeneracy
  - H Bordered Hessian det $H \neq 0$
Find h in tangent space
Check sign of $h^T H_L h$.

Example 3.3. *:*
Maximise $xyz = f(x, y, z)$. Subject to

$$g : x + y + z - 1 = 0 \tag{1}$$

Solution.

$$L(x, y, z) = xyz + \lambda(x + y + z - 1)$$
$$yz + \lambda = 0 \quad (2)$$
$$xz + \lambda = 0 \quad (3)$$
$$xy + \lambda = 0 \quad (4)$$

$$yz = xz = xy = -\lambda$$

Either $x = y$ or $z = 0$ from the first.

If $z = 0$, then $\lambda = 0$ and so $xy = 0$ and at least one of one of x, y is zero. Note $x = y = z = 0$ does not satisfy (1). Assume $x = 0$ and $y \neq 0$. Then from (1) $y = 1$ so $(0, 1, 0)$ is a solution.
By symmetry $(1, 0, 0)$ and $(0, 0, 1)$ are solutions.

If $z \neq 0$ then $x = y$. Since we have already considered the case $x = 0 = y$ we may assume $x = y \neq 0$. From equations two and three $z = y$ so $x = y = z$..

Substitute on constraint: $3x = 1$

$$\Rightarrow x = y = z = 1/3 \qquad \text{and } \lambda = -1/9.$$

check sufficiency at the critical point. $(1/3, 1/3, 1/3)$ with

$\lambda = -1/9$.

$$L = xyz + \lambda(x + y + z - 1)$$
$$H_L = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 \end{bmatrix}$$
$$\nabla g = \operatorname{grad} g = [1, 1, 1]^T$$
$$H = \begin{bmatrix} H_L & \nabla g \\ \nabla g^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 & 1 \\ 1/3 & 0 & 1/3 & 1 \\ 1/3 & 1/3 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Noting $C_2 - > C_2 - C_1$, $C_3 - > C_3 - C_1$, where C_r stands for column r, does not change the determinant we obtain

$$\det H = \begin{vmatrix} 0 & 1/3 & 1/3 & 1 \\ 1/3 & -1/3 & 0 & 1 \\ 1/3 & 0 & -1/3 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} \neq 0$$

Hence (1/3, 1/3, 1/3) is a nondeg. crit. pt.
Find h on tangent space to g at a

$$h^T \nabla g = 0, \quad [h_1\, h_2\, h_3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$
$$\Rightarrow h_1 + h_2 + h_3 = 0$$
$$h_1 = \gamma, \quad h_2 = \mu, \quad h_3 = -\gamma - \mu, \qquad \text{all } \gamma, \mu \in \mathbb{R}$$

Check

$$
\begin{aligned}
& h^T H_L h \\
&= \begin{pmatrix} \gamma & \mu & -(\gamma+\mu) \end{pmatrix} \begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \mu \\ -(\gamma+\mu) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\max &= \begin{pmatrix} \gamma & \mu & -\gamma-\mu \end{pmatrix} \begin{pmatrix} -\frac{\gamma}{3} \\ -\frac{\mu}{3} \\ \frac{\gamma+\mu}{3} \end{pmatrix} \\
&= \frac{-\gamma^2}{3} - \frac{\mu^2}{3} - \frac{(\gamma+\mu)^2}{3} < 0
\end{aligned}
$$

$\blacksquare$

Finish above **Example 3.2**:

$$
\begin{aligned}
\text{Maximise} \quad & f: \; x_1^3 + x_2^3 + x_3^3 \\
\text{Constraints} \quad & g_1: \; x_1^2 + x_2^2 + x_3^2 - 5 = 0 \\
& g_2: \; x_1^2 + x_2^2 + x_3^2 - 2x_1 - 3 = 0.
\end{aligned}
$$

We had found the critical points

$$
\begin{aligned}
(1,0,2), \quad & \lambda_1 = -3/2, \quad \lambda_2 = -3/2 \\
(1,0,-2), \quad & \lambda_1 = -3/2, \quad \lambda_2 = 9/2 \\
(1,2,0), \quad & \lambda_1 = -3/2, \quad \lambda_2 = -3/2 \\
(1,-2,0), \quad & \lambda_1 = -3/2, \quad \lambda_2 = 9/2
\end{aligned}
$$

$$L = x_1^3 + x_2^3 + x_3^3 + \lambda_1(x_1^2 + x_2^2 + x_3^2 - 5)$$
$$+\lambda_2(x_1^2 + x_2^2 + x_3^2 - 2x_1 - 3)$$

$$H_L = \begin{bmatrix} 6x_1 + 2\lambda_1 + 2\lambda_2 & 0 & 0 \\ 0 & 6x_2 + 2\lambda_1 + 2\lambda_2 & 0 \\ 0 & 0 & 6x_3 + 2\lambda_1 + 2\lambda_2 \end{bmatrix}$$

$$\nabla g_1 = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} 2x_1 - 2 \\ 2x_2 \\ 2x_3 \end{bmatrix}.$$

Check, for each of the critical points, if

$$H = \begin{bmatrix} H_L & \nabla g_1 & \nabla g_2 \\ \nabla g_1^T & & \\ \nabla g^T & 0 & \end{bmatrix}$$

has nonzero determinate.

Check $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\lambda_1 = \lambda_2 = -3/2$.

$$H = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & 4 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

$\det H = -2 \times 6 \times (-4) \times (-2) \times (-16) \neq 0$ So $(1, 0, 2)$ is non-degenerate

\# Find $\underset{\sim}{h}$ satisfying

$$\underset{\sim}{h}^T \nabla g_1 = 0 \quad \text{at crit. point}$$
$$\underset{\sim}{h}^T \nabla g_2 = 0$$

$$(h_1\, h_2\, h_3) \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 0 \quad \text{and} \quad [h_1\, h_2\, h_3] \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 0$$

$$\left.\begin{aligned} 2h_1 + 4h_3 &= 0 \\ 4h_3 &= 0 \end{aligned}\right\} h_1 = h_3 = 0, \quad h_2 = \mu$$

$$\underset{\sim}{h}^T = [0, \mu, 0]. \qquad \mu \in \mathbb{R}$$

\# Check sign of $h^T H_L h$

$$[0\, \mu\, 0] \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ \mu \\ 0 \end{bmatrix} = -6\mu^2 < 0.$$

Hence this critical point is local maximum.

Repeat the procedure for other critical points.

Revision of ODEs

\# Separable 1st order

\# Constant Coefficient 2nd order

- Homogeneous
- Nonhomogeneous

\# Separable

$$\dot{x} = f(t)g(x)$$
$$\int \frac{dx}{g(x)} = \int f(t)dt + c.$$

EXAMPLES (1)

$$\frac{dx}{dt} = -xt$$
$$\int \frac{dx}{x} = -\int t\,dt\; + c$$
$$ln\,x = -t^2/2 + c$$
$$x = Ae^{-t^2/2}$$

(2)

$$
\begin{aligned}
dx/dt &= (1+x^2)/(1-t^2) \\
\int \frac{dx}{1+x^2} &= \int \frac{dt}{1-t^2} + c \\
\arctan x &= \int \left(\frac{1/2}{1-t} + \frac{1/2}{1+t} \right) dt + c \\
&= -\frac{1}{2} ln(1-t) + \frac{1}{2} ln(1+t) + c \\
&= \left[\frac{1+t}{1-t} \right]^{1/2} + c \\
x &= \tan \left[\left(\frac{1+t}{1-t} \right)^{1/2} + c \right].
\end{aligned}
$$