MATH3404

Translating *time* so that P_1O reaches 0 at $t = 0$; if $S = 0$ there, the previous switch was at $y = -\tau$, at P_1. If $S \neq 0$ at 0, the previous switch must have been at some point Q. Trace back by π along the $\mathcal{C}^-$ curve through Q. Curve "expands" by a factor $e^{k\pi}$.

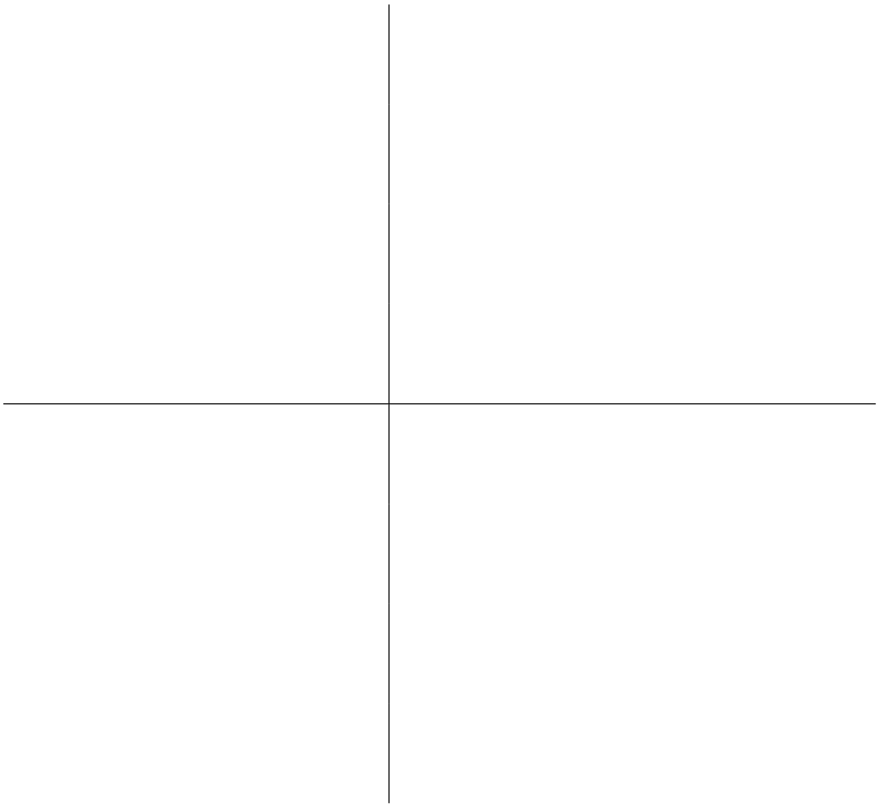
Can show that the locus of R is

$$\begin{aligned} x_1 &= -1 - 2e^{k\pi} + e^{k(\pi-\sigma)}\cos\sigma \\ x_2 &= e^{k(\pi-\sigma)}\sin\sigma \qquad -\pi \leq \sigma \leq 0. \end{aligned}$$

Each of the arcs of the switching curve is magnified by the factor $e^{k\pi}$ and translated by stretched amounts.

$$u^* = \begin{cases} -1 & \text{above } S = 0 \text{ \& on } P_2O \\ +1 & \text{below } S = 0 \text{ \& on } P_1O. \end{cases}$$

Remark: The case for $Re(\lambda) > 0$ is similar, except the loops get smaller.



$$\text{Arcs}\ \ C_1^+\ : x_1 = 1 - e^{+k\sigma}\cos\sigma,\ x_2 = e^{k\sigma}\sin\sigma$$

$$-\pi \leq \sigma \leq 0$$

$$C_2^+\ : x_1 = 1 + 2e^{-k\pi} - e^{-k(\pi-\sigma)}\cos\sigma,$$

$$x_2 = e^{-k(\pi-\sigma)}\sin\sigma$$

$$\vdots$$

$$x_2 = e^{-k(\pi-\sigma)}\sin\sigma.$$

Distance between P_n & P_{n+1} is

$$e^{-nk\pi}(1 + e^{-k\pi}) \to 0 \quad \text{as } n \to \infty.$$

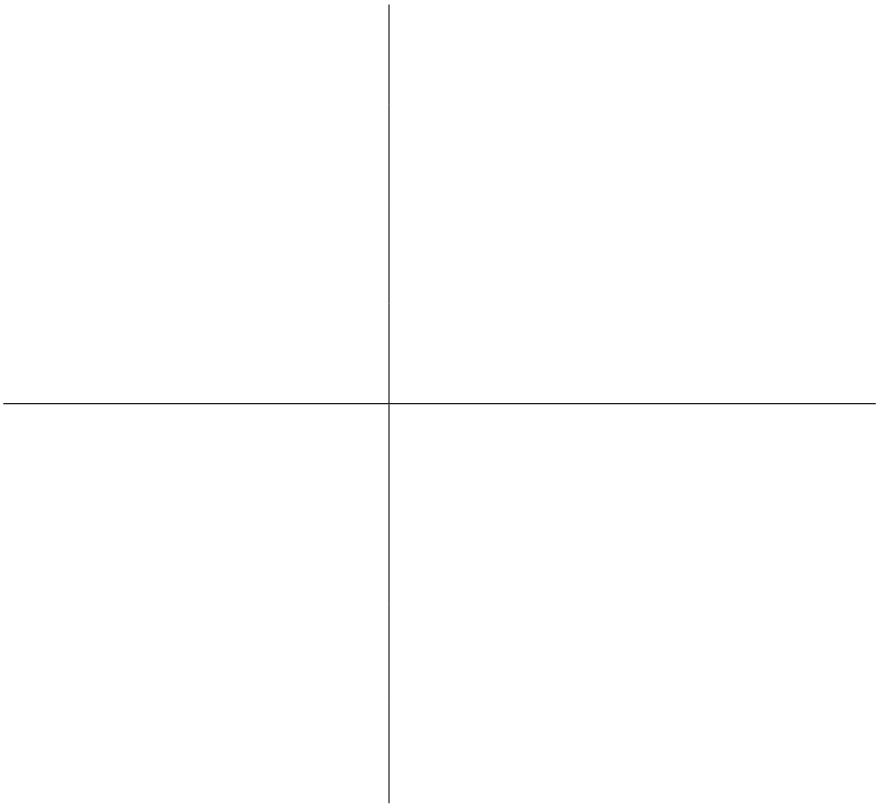
Hence

$$
\begin{aligned}
OP_{n+1} &= 1 + 2\sum_{p=1}^{\infty} e^{-pk\pi} + e^{-nk\pi} \\
&\to 1 + 2\sum_{1}^{\infty} e^{-pk\pi} \\
&= \frac{1+e^{-k\pi}}{1-e^{-k\pi}}
\end{aligned}
$$

as $n \to \infty$ similarly for OQ_{n+1}.

Although there are an infinite number of arcs, the switching curve is bounded.

$$
u^* = \begin{cases} -1 & \text{above } \Gamma \text{ \& on } \mathcal{C}_1^- \\ +1 & \text{below } \Gamma \text{ \& on } \mathcal{C}_1^+ \end{cases}
$$



Next topic:

J involves $\underset{\sim}{x}(t_1)$ the final state.

Pontryagin Max Princ for Control to a Target Curve $\mathcal{C}$.

$$
\begin{aligned}
\dot{x}_1 &= f_1(\underset{\sim}{x}, u)\,, \quad \dot{x}_2 = f_2(\underset{\sim}{x}, u) \\
J &= \int_{t_0}^{t_1} f_0(\underset{\sim}{x}, u)dt
\end{aligned}
$$

Theorem. $u^*(t)$ an admissible control taking $\underset{\sim}{x}^0$ at t_0 to a point on $\mathcal{C}: \ G(x_1,\, x_2) = 0$ at $t = t_1$. For u^*, $\underset{\sim}{x}^*$ optimal, it is necessary that $\exists\, \underset{\sim}{\psi}$, $\dot{\psi}_i = -\partial H/\partial x_i$, $i = 1,\, 2$, where

$$H = -f_0 + \psi_1 f_1 + \psi_2 f_2$$

such that

- H maximized at $u = u^*(t)$ for each $t_0 \leq t \leq t_1$
- $H(\underset{\sim}{\psi}^*, \underset{\sim}{x}^*, u^*) = 0$ (since final time t_1 is unspecified)

- $\begin{pmatrix} \psi_1(t_1) \\ \psi_2(t_1) \end{pmatrix}$ perpendicular to tangent at $\mathcal{C}$ at $\underset{\sim}{x}^*(t_1) = (x_1^*(t_1),\, x_2^*(t_1))$ Transversality condition

Corollary. *If the state* $\underset{\sim}{x}^1$ *at* $t = t_1$ *is completely unspecified, the transversality conditon becomes*

$$\begin{pmatrix} \psi_1(t_1) \\ \psi_2(t_1) \end{pmatrix} = \underset{\sim}{0}.$$

Transversality condition at Q is

(†) $\quad a\psi_1(t_1) + b\psi_2(t_1) = 0$, where $\binom{a}{b}$ tangent to $G(\underset{\sim}{x}) = 0$ at $\underset{\sim}{x} = \underset{\sim}{x}^*(t_1)$. If the final state is completely unspecified, (†) holds for all curves – **i.e.** all a, b. Hence $\psi_1(t_1) = \psi_2(t_1) = 0$.

In particular, if the system is governed by a single DE with free endpoint $x(t_1)$, transversality condition is just

$$\psi_1(t_1) = 0.$$

Problems where cost depends on $\underset{\sim}{x}(t_1)$.

Realistic costs often involve the final state of a system. For example, in a medical control problem, we may be trying to maximize the concentration of a drug; or in an industrial process perhaps trying to minimize the quantity of some by final by product which is a pollutant.

Moreover, if we wished to have controls *without constraints*, then to prevent a solution with unbounded controls, we might introduce a heavy penalty, using terms like $\int_{t_0}^{t_1} u^2 dt$ and obtain

$$J = -x(t_1) + \int_{t_0}^{t_1} u^2 dt.$$

General Problem

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}) = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Control the system in $t_0 \leq t \leq t_1$ from $\underset{\sim}{x}^0$ at $t = t_0$ to $\underset{\sim}{x}^1$ at $t = t_1$ in such a way that

$$J = g(\underset{\sim}{x}^1) + \int_{t_0}^{t_1} f_0(\underset{\sim}{x}, \underset{\sim}{u})dt$$

is minimized. Find the optimal control.

As stated, $\underset{\sim}{x}^1 = \underset{\sim}{x}(t_1)$ is free – the transversality condition will have to be used. # Introduce a new cost variable X_0,

$$\begin{aligned} \dot{X}_0 &= \textstyle\sum_1^m \frac{\partial g}{\partial x_i} f_i + f_0 \\ &= \textstyle\sum_1^m \frac{\partial g}{\partial x_i} \dot{x}_i + f_0 \end{aligned} \qquad X_0(t_0) = 0$$

$$\Rightarrow \quad X_0(t_1) - X_0(t_0) = g(\underset{\sim}{x}(t_1)) - g(\underset{\sim}{x}(t_0)) + \int_{t_0}^{t_0} f_0 dt$$

Since $X_0(t_0) = 0$, we have,

$$X_0(t_1) = J - g(\underset{\sim}{x}(t_0)).$$

Since $\underset{\sim}{x}(t_0) = \underset{\sim}{x}^0$ is given, the quantity $g(\underset{\sim}{x}(t_0))$ is known and constant. Hence, minimizing X_0 is the same as minimizing J.

- Transversality conditions (C of V)

- Complex eigenvalues switching curve

- Riccati equation

Apply Pontryagin Max Princ:

$$\begin{aligned} H &= \psi_0 \dot{X}_0 + \psi_1 f_1 + \cdots + \psi_n f_n \\ &= \not{\psi}_0 \left\{ f_0 + \sum_1^n \frac{\partial g}{\partial x_j} f_j \right\} + \sum_1^n \psi_j f_j . \end{aligned}$$

As previously, take $\psi_0 = -1$ and the costate equations are

$$\dot{\psi}_i = -\partial H / \partial x_i \ , \quad i = 1, ..., n.$$

However, these are more complicated than the case where J does not involve $\underset{\sim}{x}(t_1)$. Let's look for a simplification.

Rearrange H

$$H = -f_0 + \sum_1^n \left[\psi_j - \partial g / \partial x_j \right] f_j .$$

Introduce "pseudo-costate" variables

$$\begin{aligned} \lambda_i &= \psi_i - \partial g / \partial x_i \ , \quad i = 1, 2, ..., n \\ H &= -f_0 + \sum_{1}^{n} \lambda_j f_j := H'. \end{aligned}$$

It turns out that

$$\dot{\lambda}_i = -\frac{\partial H'}{\partial x_i} \ , \quad i = 1, ..., n$$

* The λ_i's formally act like costate variables and the equations are much easier to solve.

Since $\underset{\sim}{x}(t_1) = \underset{\sim}{x}^1$ free, so the transversality condition is

$$\begin{aligned} \psi_i(t_1) = 0 \, , \quad i = 1, ..., n. \\ \Rightarrow \quad \lambda_i(t_1) = -\tfrac{\partial g}{\partial x_i}(t_1) \, , \, i = 1, ..., n. \end{aligned}$$

Summary:

- Write $H' = -f_0 + \sum_j \lambda_j f_j$
- Maximize H' as a function of u.

- End conditions $\underset{\sim}{x}(t_0) = \underset{\sim}{x}^0$

  Two endpoint

  Boundary Value $\qquad \lambda_i(t_1) = -\left.\frac{\partial g}{\partial x_i}\right|_{t=t_1}$

  Problem.

Example. $\dot{x} = -\alpha x + u$, controlled from $x = 0$ at $t = 0$ to $x(t_1)$ at a fixed time t_1, minimizing

$$J = -x(t_1) + \int_0^{t_1} u^2 dt.$$

Find the optimal control u^*.

(Control u is unconstrained, but the u^2 term makes it expensive to use too much.)

Solution.

$$H = H' = \overset{-f_0}{-u^2} + \overset{\lambda_1 f_1}{\lambda(-\alpha x + u)}$$

Costate equations: $\dot{\lambda} = -\partial H / \partial x$

$$\dot{\lambda} = \alpha\lambda\,, \quad \lambda = Ae^{\alpha t}$$

To maximize H' as a function of u,

$$H'_u = -2u + \lambda = 0 \qquad \boxed{u^* = \lambda/2}$$

So $u^* = Ae\frac{\alpha t}{2}$.

Optimal state equation

$$\begin{aligned} \dot{x} &= -\alpha x + u^* = -\alpha x + Ae^{\alpha t}/2 \\ \Rightarrow \qquad x &= Be^{-\alpha t} + Ae^{\alpha t}/4\alpha \end{aligned}$$

End conditions:

$$x(0) = 0$$

$$0 = B + A/4\alpha\,, \quad B = -A/4\alpha$$

$$\text{At} \quad t = t_1\,, \quad \lambda = -\tfrac{\partial g}{\partial x}$$

$$\text{Now} \quad g(x(t_1)) = -x(t_1)\,, \quad \text{so } g(x) = -x$$

$$\Rightarrow \quad \lambda(t_1) = -\tfrac{\partial g}{\partial x} = +1\,, \quad Ae^{\alpha t_1} = +1$$

$$A = +e^{-\alpha t_1}$$

Hence $u^* = +\frac{e^{-\alpha t_1}}{2}e^{\alpha t} = +\frac{1}{2}e^{\alpha(t-t_1)}$.

The optimal trajectory is

$$\begin{aligned} x &= -e^{-\alpha t_1}e^{-\alpha t} + \frac{1}{4\alpha}e^{\alpha(t-t_1)} \\ &= \frac{e}{2\alpha}\left[\frac{e^{\alpha(t)} - e^{-\alpha(t)}}{2}\right] \\ &= \frac{e^{-\alpha t_1}}{2\alpha}\sinh \alpha t. \end{aligned}$$

This example we have just done is close to a special case of a wide class of useful and realistic systems: