Linear equation - Quadratic cost

(1) $\dot{x} = A(t)\,x + B(t)\,u$, A cts real
$x(t_0) = x_0$ $\quad$ B cts real

(2) $J = \frac{1}{2} S x^2(t_1) + \frac{1}{2}\int_{t_0}^{t_1} (Px^2 + 2Qxu + Ru^2)\,dt$

t_0, t_1 fixed $\quad x(t_1)$ free

$$H' = -f_0 + \lambda f_1$$
$$= -\tfrac{1}{2}(Px^2 + 2Qxu + Ru^2) + \lambda(Ax + Bu)$$

u unconstrained so H' maximised

when $\frac{\partial H'}{\partial u} = -Qx - Ru + \lambda B = 0$ (3)

iff $R > 0$

Assume $R > 0$

(3) $\Rightarrow$ $u^* = -R^{-1}Qx + R^{-1}B\lambda$ (4)

Costate $\quad \dot{\lambda} = -\frac{\partial H'}{\partial x}$

$$\dot{\lambda} = +Px + Qu - \lambda A \quad (5)$$

State $\quad \dot{x} = Ax + Bu \quad (1)$

Using (4)

(*)
$$\begin{cases} \dot{x} = Ax + B(-R^{-1}Qx + R^{-1}B\lambda) \\ \quad = (A - BR^{-1}Q)x + BR^{-1}B\,\lambda \\ \dot{\lambda} = Px + Q(-R^{-1}Qx + R^{-1}B\lambda) - \lambda A \\ \quad = (P - QR^{-1}Q)x + (QR^{-1}B - A)\lambda \end{cases}$$

$x(t_1)$ free

$$\Rightarrow \lambda(t_1) = \psi(t_1) - \frac{\partial g}{\partial x}$$

$$\Leftrightarrow \left.\begin{aligned} \lambda(t_1) &= -Sx(t_1) \\ x(t_0) &= x_0 \end{aligned}\right\} \text{Boundary Value Problem} \qquad (6)$$

Let $y = \begin{pmatrix} x \\ \lambda \end{pmatrix} \Rightarrow \dot{y} = D(t)\, y \qquad y(t_1) = y_1 = \begin{pmatrix} x(t_1) \\ \lambda(t_1) \end{pmatrix} \qquad (7)$

Let $\dot{\varphi}_1 = D(t)\, \varphi_1 \qquad \varphi_1(t_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\dot{\varphi}_2 = D(t)\, \varphi_2 \qquad \varphi(t_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Let $\Phi(t, t_1) = (\varphi_1 \,\vdots\, \varphi_2)$

$$\Rightarrow \dot{\Phi} = D(t)\, \Phi \qquad (8)$$

Solution of (7) is

$$y(t) = \Phi(t, t_1)\, y_1$$

$$\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} F(t,t_1) & G(t,t_1) \\ L(t,t_1) & M(t,t_1) \end{pmatrix} \begin{pmatrix} x(t_1) \\ \lambda(t_1) \end{pmatrix}$$

$$x(t) = F\, x(t_1) + G\, \lambda(t_1) \qquad (9)$$

$$\lambda(t) = L\, x(t_1) + M\, \lambda(t_1) \qquad (10)$$

Now $(6) \Rightarrow \lambda(t_1) = -Sx(t_1)$

$$(9) \Rightarrow x(t) = (F - GS)\, x(t_1) \qquad (11)$$

$$(10) \Rightarrow \lambda(t) = (L - MS)\, x(t_1) \qquad (12)$$

(11) $\Rightarrow$ $x(t_1) = (F - GS)^{-1} x(t)$

(12) $\Rightarrow$ $\lambda(t) = (L - MS)(F-GS)^{-1} x(t)$

(13) $\lambda(t) = K(t, t_1) x(t)$

$\dot{\lambda} = \dot{K} x + K \dot{x}$

use (*) to sub for $\dot{\lambda}$ and $\dot{x}$

$$\Rightarrow \underbrace{[\dot{K} + (KB - Q)R^{-1}(BK - Q) + KA + AK - P]}_{=0} x(t) = 0$$

(14) $\dot{K} = -(KB-Q)R^{-1}(BK-Q) - KA - AK + P$

Now $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi(t_1, t_1) = \begin{pmatrix} F(t_1,t_1) & G(t_1,t_1) \\ L(t_1,t_1) & M(t_1,t_1) \end{pmatrix}$

(15) $\Rightarrow$ $F(t_1) = 1 = M(t_1)$ $\quad G(t_1) = 0 = L(t_1)$

(13) $\Rightarrow$ $-Sx(t_1) = \lambda(t_1) = K(t_1) x(t_1)$

(16) $\Rightarrow$ $K(t_1) = -S$

Solve (14) & (16) for K then use (13) & (4)

$\Rightarrow$ $u^* = R^{-1}(BK - Q)x$

Sub in state equation and solve !!

Can do same for systems - A $n \times n$ B $n \times m$ etc - get matrix equations & some transposes thrown in!

Application of Riccati

State eqn $\dot{x} = x + u$

Control from $x(0) = x_0$ minimizing

$$J = \frac{1}{2}x_1^2 + \frac{1}{2}\int_0^1 u^2 dt$$

where $x_1 = x(1)$

$$H' = -\frac{1}{2}u^2 + \lambda(x+u)$$

$$\frac{\partial H'}{\partial u} = -u + \lambda, \quad \frac{\partial^2 H'}{\partial u^2} = -1 < 0$$

so H' maximised when $\frac{\partial H'}{\partial u} = 0$, when $u = \lambda$

$$\dot{\lambda} = -\frac{\partial H'}{\partial x} = -\lambda \Rightarrow \lambda = Ae^{-t}$$

$$\lambda(1) = -\frac{\partial g}{\partial x_1} = -x_1$$

Set $\lambda = kx \Rightarrow \dot{\lambda} = \dot{k}x + k\dot{x}$

$$-\lambda = \dot{k}x + k(x+u)$$

but $\lambda = kx$ and $u = \lambda$

$$\Rightarrow -kx = \dot{k}x + kx + k^2x$$

$$\dot{k} = 2k + k^2$$

$$\frac{1}{2}\left(\int \frac{1}{k} - \frac{1}{k+2}\right)dk = \int \frac{dk}{2k+k^2} = t + l \qquad k(1) = -1$$

$$\ln\left|\frac{k}{k+2}\right| = 2t + l$$

$$\frac{-k}{k+2} = e^{-2t+l} \qquad k(1) = -1$$

$$\Rightarrow 1 = e^{-2+l} \Rightarrow l = 2$$

$$-\frac{K}{2} = \frac{e^{-2(t-1)}}{1+e^{-2(t-1)}}$$

Now $\lambda = Kx$ so optimal state eqn

$$\dot{x} = x+u = x+\lambda = x+Kx \quad , x(0)=x_0$$

$$= \frac{1-e^{-2(t-1)}}{1+e^{-2(t-1)}}\, x$$

$$= \tanh[2(t-1)]\, x$$

$$\ln|x| = \frac{1}{2}\ln\cosh[2(t-1)] + d$$

Case $x_0 > 0$

$$x = A\cosh^2 2(t-1)$$

$$= \frac{x_0}{\cosh^2 2}\cosh^2 2(t-1)$$

$$u^* = \lambda = Kx \quad \text{etc}$$

Quick Calculus of Variations derivation of part of Pontryagin u not constrained

$$J = g_0(t_0, x_0, t_1, x_1) + \int_{t_0}^{t_1} f_0 \, dt$$

$$\dot{x} = f(t, x, u) \quad , \quad g(t_0, x_0, t_1, x_1) \equiv 0$$

$$\Rightarrow \quad J = g_0 + \int_{t_0}^{t_1} (-\mathcal{H} + \psi \dot{x}) \, dt$$

$$= g_0 + \int_{t_0}^{t_1} \mathcal{L} \, dt$$

where $\mathcal{H} = -f_0 + \psi(\dot{x} - f)$ & $\mathcal{L} = -\mathcal{H} + \psi \dot{x}$

For x, u to minimize J

ψ, x, u must minimize J with fixed end point problem $\Rightarrow$ $\mathcal{L}$ must satisfy E E

$$0 = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}} = \frac{\partial \mathcal{L}}{\partial u} = -\frac{\partial \mathcal{H}}{\partial u}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}}{\partial \psi} = \dot{x} - f \quad \text{(state eqn)}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{\partial \mathcal{H}}{\partial x} - \frac{d}{dt}\psi \quad \text{(costate eqns)}$$

Optimal Control – a Calculus of Variations approach – scalar case

$$J = g_0(t_0, x_0, t_1, x_1) + \int_{t_0}^{t_1} f_0(t, x, u)\,dt \quad (1)$$

$u(t) \in \mathcal{U}(t) \qquad t_0 \le t \le t_1$

(2) $\dot{x} = f(t, x, u) \qquad t_0 \le t \le t_1$

(3) $g_i(t_0, x_0, t_1, x_1) = 0 \qquad 1 \le i \le 4, \; x(t_i) = x_i$

Find $u^*(t)$, $x^*(t)$ such that

(4) $J(u^*, x^*) \le J(u, x)$

* Some notation: If $j = j(t, x(t), u(t))$

(5) $j(t) = j(t, x(t), u(t)) = j(x, u)$

$$\Delta j = j(x+\delta x, u+\delta u) - j(x, u)$$
$$= j(x+\delta x, u+\delta u) - j(x, u+\delta u) + j(x, u+\delta u) - j(x, u)$$

(6) $\Delta j = j_x(x, u)\,\delta x + j(x, u+\delta u) - j(x, u) + o(\delta x, \delta u)$

by Taylor where $\dfrac{o(\delta x, \delta u)}{|\delta t_0| + |\delta t_1| + |\delta x| + |\delta u|} \to 0$ as $\begin{pmatrix} |\delta x| + |\delta u| \\ |\delta t_0| + |\delta t_1| \end{pmatrix} \to 0$

(**) If $I(x, u) = \int_{t_0}^{t_1} j(x, u)\,dt$

$$\Delta I = I(x+\delta x, u+\delta u) - I(x, u)$$
$$= \int_{t_0+\delta t_0}^{t_1+\delta t_1} j(x+\delta x, u+\delta u)\,dt - \int_{t_0}^{t_1} j(x, u)\,dt$$
$$= \left(\int_{t_1}^{t_1+\delta t_1} - \int_{t_0}^{t_0+\delta t_0}\right) j(x+\delta x, u+\delta u)\,dt + \int_{t_0}^{t_1} \Delta j\,dt$$

$$\Delta I = f(t_1)\,\delta t_1 - f(t_0)\,\delta t_0 + \int_{t_0}^{t_1} f_x(x,u)\,\delta x\,dt + \int_{t_0}^{t_1}\left(\begin{array}{l} f(x,u+\delta u) \\ -f(x,u)\end{array}\right)dt$$
(7)
$$+\, o(\delta x, \delta u)$$

Let
$$\tilde g = g_0 + \sum_{i=1}^{4} a_i g_i \tag{8}$$
$$\Phi(x,u) = \int_{t_0}^{t_1} \psi(t)(\dot x - f(x,u))\,dt \equiv 0 \tag{9}$$
$$I(x,u) = \int_{t_0}^{t_1} f_0(x,u)\,dt \tag{10}$$

$$J = \tilde g + I + \Phi \tag{11}$$
$$\Delta J = \Delta \tilde g + \Delta I + \Delta \Phi \tag{12}$$
$$\Delta \tilde g = \tilde g_{t_0}\,\delta t_0 + \tilde g_{t_1}\,\delta t_1 + \tilde g_{x_0}\,\delta x_0 + \tilde g_{x_1}\,\delta x_1 + o \tag{13}$$

by (**)
$$\Delta \Phi = \int_{t_0}^{t_1}\{\psi\,\delta\dot x - \psi f_x\,\delta x - \psi(f(x,u+\delta u) - f(x,u))\}\,dt + o \tag{14}$$

Now
$$\int_{t_0}^{t_1} \psi\,\delta \dot x = \psi\,\delta x\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot\psi\,\delta x\,dt \tag{15}$$

by (**)
$$\Delta I = f_0(t_1)\,\delta t_1 - f_0(t_0)\,\delta t_0 + \int_{t_0}^{t_1} f_{0x}\,\delta x\,dt + \int_{t_0}^{t_1}(f_0(x,u+\delta u) - f(x,u))\,dt + o \tag{16}$$

Setting $H = -f_0 + \psi f$ and noting

$$\delta x_i = x(t_i + \delta t_i) + \delta x(t_i + \delta t_i) - x(t_i)$$
$$= \dot x(t_i)\,\delta t_i + \delta x(t_i) + o = f(t_i)\,\delta t_i + \delta x(t_i) + o$$

and using (12)-(16)

$$\Delta J = (\tilde g_{t_1} - H(t_1))\,\delta t_1 + (\tilde g_{t_0} + H(t_0))\,\delta t_0 + (\tilde g_{x_0} - \psi(t_0))\,\delta x_0$$
$$+ (\tilde g_{x_1} + \psi(t_1))\,\delta x_1 + \int_{t_0}^{t_1}(-H - \dot\psi)\,\delta x\,dt + \int_{t_0}^{t_1}(-H(x,u+\delta u) + H(x,u))\,dt$$
$$+ o$$

For a min $\Delta J \geq 0 \Rightarrow -H - \dot\psi = 0$ (costate)

if $\delta t_1 \neq 0$ $\quad \tilde g_{t_1} - H(t_1) = 0$

if $\delta t_0 \neq 0$ $\quad \tilde g_{t_0} + H(t_0) = 0$

If x_0 not fixed $\Rightarrow \tilde{g}_{x_0} - \psi(t_0) = 0$

If x_1 not fixed $\Rightarrow -\tilde{g}_{x_1} - \psi(t_1) = 0$

$$\Rightarrow \Delta J = \int_{t_0}^{t_1} \left(-\mathcal{H}(x, u+\delta u) + \mathcal{H}(x,u) \right) dt + 0 \geq 0$$

$$\Rightarrow \quad \mathcal{H}(x,u) \geq \mathcal{H}(x, u+\delta u)$$

that is, u maximizes $\mathcal{H}$.

<u>Remark</u> If $\underset{\sim}{x}(t) \in \mathbb{R}^n$ (and $g_i(t_0, x_0, t_1, x_1) = 0$, $1 \le i \le m$) set $\tilde{g} = g_0 + \sum_1^m a_i g_i$

and if x_{0k} not fixed $\Rightarrow \tilde{g}_{x_{0k}} - \psi_k(t_0) = 0$

if x_{1k} not fixed $\Rightarrow \tilde{g}_{x_{1k}} + \psi_k(t_1) = 0$

$\mathcal{H} = -f_0 + \sum_1^n \psi_i f_i$ where $\dot{x}_i = f_i(t, \underset{\sim}{x}, \underset{\sim}{u})$

$-\mathcal{H}_{x_i} = \dot{\psi}_i$ costate eqns

If $\int_{t_0}^{t_1} \varphi_s(t, x, u)\, dt = 0 \qquad 1 \le s \le p$

replace f_0 by $f_0 + \sum_{s=1}^p \eta_s \varphi_s$ where η_s are constants

<u>Remark</u> Case u unconstrained $\frac{\partial \mathcal{H}}{\partial u} = 0$

Let $M(t,x,u) = \mathcal{H}(t,x,u)$ where u and x are optimal

$$\frac{d}{dt} M = \mathcal{H}_t + \mathcal{H}_x \frac{dx}{dt} + \mathcal{H}_u \frac{du}{dt} + \mathcal{H}_\psi \frac{d\psi}{dt}$$

$$= \mathcal{H}_t - \dot{\psi} f + 0 + f \dot{\psi}$$

$$= \mathcal{H}_t \text{ so if } \mathcal{H}_t \equiv 0 \Rightarrow M \equiv \text{const}$$