Example 4.

\[ \dot{x}_1 = x_2 + u \quad , \quad |u| \leq 1, \]
\[ \dot{x}_2 = -x_2 + u \]

Here, \( \text{det} A = \text{det} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = 0 \), so there is not an isolated singularity at \( 0 \), but all of \( x_2 = 0 \). Here there is no eigenvalue/eigenvector problem.

\[ H = -1 + \psi_1 (x_2 + u) + \psi_2 (-x_2 + u) \]
\[ = -1 + \psi_1 x_2 - \psi_2 x_2 + (\psi_1 + \psi_2) u \]

Maximized for \( u^* = \text{sgn}(\psi_1 + \psi_2) = \pm 1 \).

Costate equations

\[ \dot{\psi}_1 = -\frac{\partial H}{\partial x_1} = 0 \quad \Rightarrow \quad \psi_1 = k \]
\[ \dot{\psi}_2 = -\frac{\partial H}{\partial x_2} = -\psi_1 + \psi_2 = -k + \psi_2 \]
\[ \Rightarrow \quad \psi_2 = le^t + k. \]
Switching curve $S = \psi_1 + \psi_2 = le^t + 2k$. At most one zero; that is, at most one switch.

State equations for optimal orbits:

\[
\dot{x}_1 = x_2 + u^*
\]

\[
\dot{x}_2 = -x_2 + u^*, \quad u^* = \pm 1
\]

\[
\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{x_1} = \frac{-x_2 + u^*}{x_2 + u^*}.
\]

This is zero on $x_2 = u^*$ and infinite on $x_2 = -u^*$.

On $x_2 = 0$, $\frac{dx_2}{dx_1} = 1$.

Also, $x_2 = u^*$

\[
\Rightarrow \quad \frac{dx_2}{dx_1} = 0
\]

\[
\Rightarrow \quad \text{this is a trajectory of the system}
\]
\( u^* = +1 \)

\( u^* = -1 \)
\[ u^* = +1 \]

\[ u^* = \begin{cases} 
-1 & \text{above (to the right of)} \\
\Gamma^{-0}\Gamma^+ \text{ and on } \Gamma^{-0}; \\
+1 & \text{below (to left of )} \\
\Gamma^{-O}\Gamma^+ \& \text{ on } \Gamma^{+O}. 
\end{cases} \]

This concludes our look at the case of real eigenvalues.

**Systems with complex eigenvalues**

If \( A \) (system matrix), \( \dot{x} = Ax + (\begin{pmatrix} l \\ m \end{pmatrix})u \) has complex
eigenvalues, so does the costate matrix $-A^T$, $\dot{\psi} = -A^T\psi$. Controls which maximize $H$ still piecewise constant, $u^* = \pm 1$, according to the sign of $S = L\psi_1 + M\psi_2$, but it turns out that $S$ has lots of zeros. This means that more than one switch is possible.

Observe that the system without controls, $\dot{x} = Ax$, has oscillatory behaviour. The optimal control will use the oscillation to drive the system to 0.

# Real eigenvalues, don’t really need to solve the costate eqns. Lemma states there is at most one switch.

# Complex eigenvalues: must find $\psi_1$ and $\psi_n$ to get an idea of $S$ and find the switches.

- imaginary eigenvalues,
- negative real parts,
- positive real parts.
Example 1. Imaginary eigenvalues

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -x_1 + u \]

To be controlled to 0 in minimum time, with \(|u| \leq 1\).

Solution.

\[ H = -1 + \psi_1 x_2 + \psi_2 (-x_1 + u) \]
\[ = -1 + \psi_1 x_2 - \psi_2 x_1 + u \psi_2. \]

This is maximized when

\[ u^* = sgn \psi_2 \]
\[ = \pm 1 \]

Costate Eqns.

\[ \dot{\psi}_1 = \psi_2 \]
\[ \dot{\psi}_2 = -\psi_1 \quad \ddot{\psi}_2 + \psi_2 = 0. \]

\[ A^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] has eigenvalues \( \pm i = q \)

\[ S = \psi_2 = k \sin(t + l), \] \( k \) and \( l \) arbitrary constants
Zeros of $S$ at $t = n\pi - l, \ n = 0, \pm 1, \pm 2, \ldots$

State Eqns.

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + u^*, \ u^* = \pm 1
\end{align*}
$$

$$
\begin{align*}
u^* &= 1 \\
\frac{d}{dt}(x_1 - 1) &= x_2 \\
\dot{x}_2 &= -(x_1 - 1)
\end{align*}
$$

That is

$$
\begin{align*}
\xi &= a \cos(t + \alpha) \\
x_2 &= -a \sin(t + \alpha) \\
x_1 - 1 &= a \cos(t + \alpha) \\
x_2 &= -a \sin(t + \alpha) \\
(x_1 - 1)^2 + x_2 &= a^2
\end{align*}
$$
\[ \dot{x}_1 = -x_1 + 1 \]
\[ \dot{x}_2 < 0 \text{ if } x_1 > 1 \]

\[
\begin{align*}
  u^* &= -1 \\
  \frac{d}{dt}(x_1 + 1) &= x_2 \\
  \dot{x}_2 &= -(x_1 + 1) \\
  \ddot{\xi} + \dot{\xi} &= 0 \\
  x_1 + 1 &= b \cos(t + \beta) \\
  x_2 &= -b \sin(t + \beta) \\
  (x_1 + 1)^2 + x_2^2 &= b^2
\end{align*}
\]
Optimal paths consist of circles, $C^+$ for $u^* = 1$, $C^-$ for $u^* = -1$. Optimal path to zero consists of alternate arcs of $C^+$ and $C^-$ curves.

$$S = \psi_2 = k \sin(t + l)$$

Switches at $t = n\pi - l$, $n$ integer.

So the switches will be $\pi$ apart in time. In this time $C^+$ or $C^-$ sweeps out a semicircle (because

$$x_1 + 1 = b \cos(t + \beta)$$
$$x_2 = -b \sin(t + \beta)$$

which is the equation of a circle parametrized by $t 0 \leq t \leq 2\pi$).

The origin is actually reached on either $C_1^-$ or $C_1^+$

If we are on $C_1^-$ (or $C_1^+$) the optimal strategy is obviously to stay on it. Any other initial state must either
reach $C_1^+$ on a $C^-$ path, or reach $C_1^-$ on a $C^+$ path.

Suppose we have a $C^-$ path intersects $C_1^+$ at $Q$ at time $\tau$. It must have switched to $C^-$ path from a $C^+$ path.
At $R$ at the time $\tau - \pi$, a semicircle away. Similarly an optimal path switching onto $C^-$ must have had the previous switch on the lower half of the semicircle radius 1, centre $(3, 0)$. Continue to work backwards in this way, $R$ must have come from a switch $C^-$ to $C^+$ at $R'$, on a semicircle of radius 1, centred at $(5, 0)$. And so on:

$$u^* = \begin{cases} -1 & \text{above } C \text{ and on } C^-_1 \\ +1 & \text{below } C \text{ and on } C^+_1 \end{cases}$$
Eigenvalues with negative real part:

Example 2. Suppose that the uncontrolled system has a stable focus at 0.

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-k & 1 \\
-1 & -k
\end{pmatrix} x + \begin{pmatrix}
k_1 \\
1
\end{pmatrix} u, \ k > 0
\sim
|u| \leq 1
\]

Solution.

\[
H = -1 + \psi_1(-kx_1 + x_2 + ku) \\
+ \psi_2(-x_1 - kx_2 + u)
\]

\[
u^* = \pm 1 = sgn(k\psi_1 + \psi_2).
\]
Matrix $A$ has eigenvalues $\lambda = -k \pm i$

$u^* = 1$

\[
\begin{align*}
\dot{x}_1 &= -kx_1 + x_2 + ku \\
\dot{x}_2 &= -x_1 - kx_2 + u
\end{align*}
\]

Critical point

$(1, 0) = N$

\[
\begin{align*}
x_1 - 1 &= ae^{-kt} \cos(t + \alpha) \\
x_2 &= -ae^{kt} \sin(t + \alpha)
\end{align*}
\]

$u^* = -1$

Critical point

$(-1, 0) = M$

\[
\begin{align*}
x_1 + 1 &= ae^{-kt} \cos(t + \alpha) \\
x_2 &= -ae^{-kt} \sin(t + \alpha)
\end{align*}
\]