MATH3404

Example 4.

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= -x_2 + u \end{aligned} \quad , \quad |u| \le 1,$$

Here, $\det A = \det \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = 0$, so there is not an isolated singularity at $\underset{\sim}{0}$, but all of $x_2 = 0$. Here there is no eigenvalue/eigenvector problem.

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$$\begin{aligned} H &= -1 + \psi_1(x_2 + u) + \psi_2(-x_2 + u) \\ &= -1 + \psi_1 x_2 - \psi_2 x_2 + (\psi_1 + \psi_2)u \end{aligned}$$

Maximized for $u^* = sgn(\psi_1 + \psi_2) = \pm 1$.

Costate equations

$$\begin{aligned} \dot{\psi}_1 &= -\frac{\partial H}{\partial x_1} = 0 \quad \Rightarrow \quad \psi_1 = k \\ \dot{\psi}_2 &= -\frac{\partial H}{\partial x_2} = -\psi_1 + \psi_2 = -k + \psi_2 \\ \Rightarrow \quad \psi_2 &= le^t + k. \end{aligned}$$

Switching curve $S = \psi_1 + \psi_2 = le^t + 2k$. At most one zero; that is, at most one switch.

State equations for optimal orbits:

$$\dot{x}_1 = x_2 + u^*$$

$$\dot{x}_2 = -x_2 + u^* \ , \quad u^* = \pm 1$$

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 + u^*}{x_2 + u^*}.$$

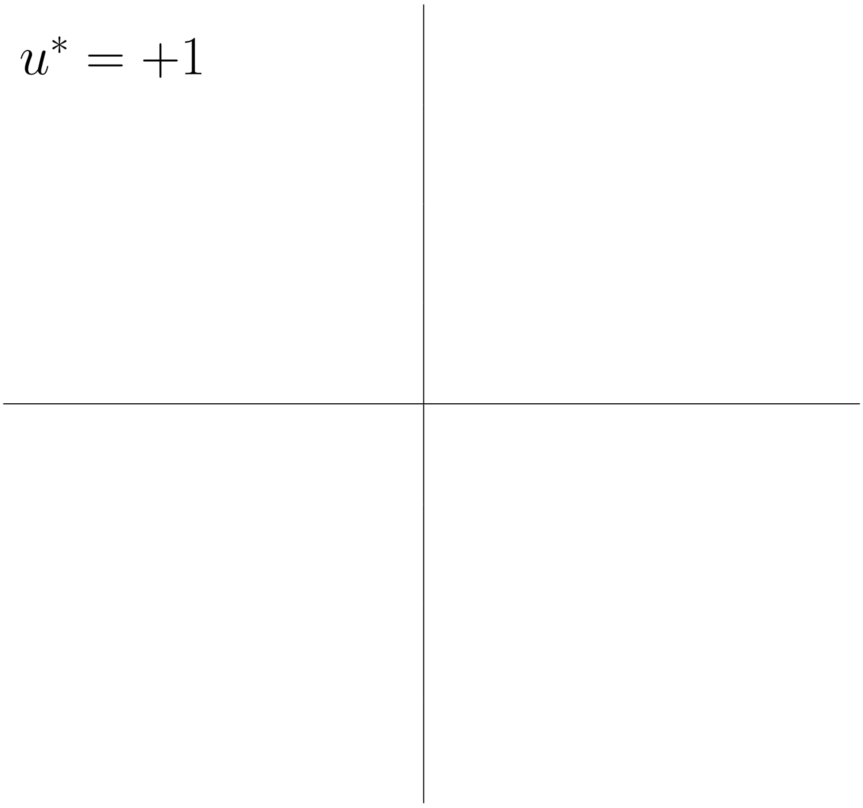
This is zero on $x_2 = u^*$ and infinite on $x_2 = -u^*$.

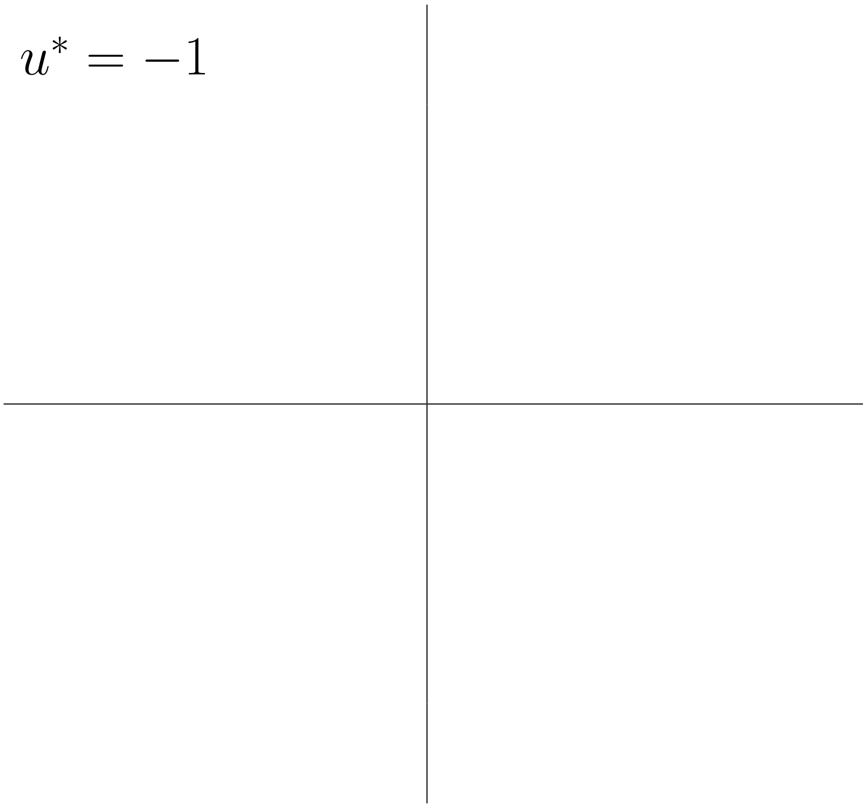
On $x_2 = 0$, $\frac{dx_2}{dx_1} = 1$.

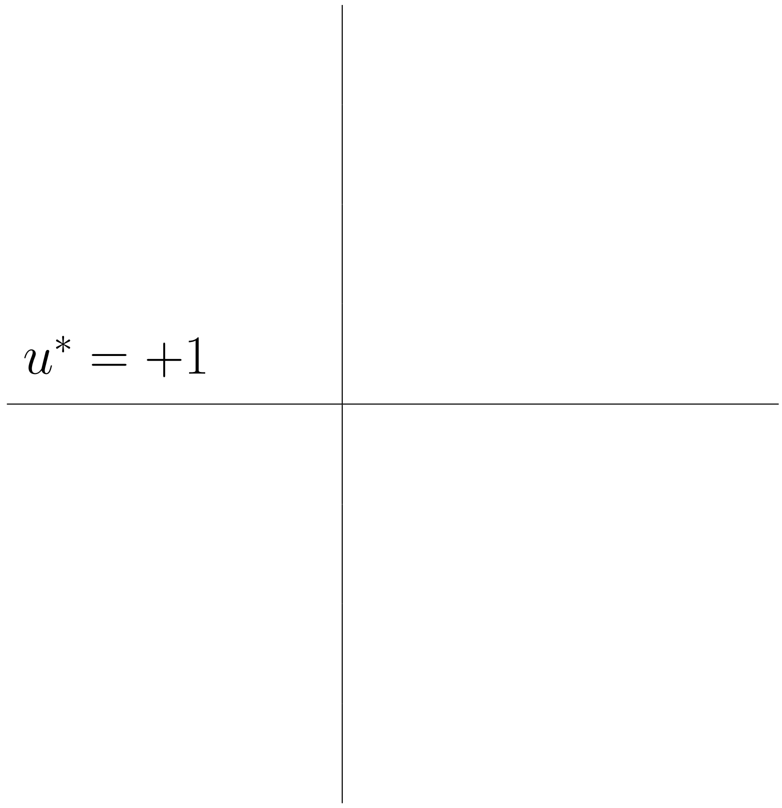
Also, $\qquad x_2 = u^*$

$$\Rightarrow \qquad \frac{dx_2}{dx_1} = 0$$

$\Rightarrow$ this is a trajectory of the system


$u^* = +1$


$u^* = -1$



$$u^* = \begin{cases} -1 & \text{above (to the right of)} \\ & \Gamma^- 0 \Gamma^+ \text{ and on } \Gamma^- 0; \\ +1 & \text{below (to left of)} \\ & \Gamma^- O \Gamma^+ \text{ \& on } \Gamma^+ O. \end{cases}$$

This concludes our look at the case of real eigenvalues.

Systems with complex eigenvalues

If A (system matrix), $\underset{\sim}{\dot{x}} = A\underset{\sim}{x} + \binom{l}{m} u$ has complex eigenvalues, so does the costate matrix $-A^T$, $\dot{\underset{\sim}{\psi}} = -A^T \underset{\sim}{\psi}$. Controls which maximize H still piecewise constant, $u^* = \pm 1$, according to the sign of $S = L\psi_1 + M\psi_2$, but it turns out that S has lots of zeros. This means that more than one switch is possible.

Observe that the system without controls, $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, has oscillatory behaviour. The optimal control will use the oscillation to drive the system to $\underset{\sim}{0}$.

Real eigenvalues, don't really need to solve the costate eqns. Lemma states there is at most one switch.

Complex eigenvalues: must find ψ_1 and ψ_n to get an idea of S and find the switches.

- imaginary eigenvalues,
- negative real parts,
- positive real parts.

Example 1. Imaginary eigenvalues

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned} \; ; \; \dot{\underset{\sim}{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

To be controlled to $\underset{\sim}{0}$ in minimum time, with $|u| \leq 1$.

Solution.

$$\begin{aligned} H &= -1 + \psi_1 x_2 + \psi_2(-x_1 + u) \\ &= -1 + \psi_1 x_2 - \psi_2 x_1 + u\psi_2. \end{aligned}$$

This is maximized when

$$\begin{aligned} u^* &= sgn\, \psi_2 \\ &= \pm 1 \end{aligned}$$

Costate Eqns.

$$\begin{aligned} \dot{\psi}_1 &= \psi_2 \\ \dot{\psi}_2 &= -\psi_1 \qquad \ddot{\psi}_2 + \psi_2 = 0. \end{aligned}$$

$A^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\pm i = q$

$S = \psi_2 = k \sin(t + l)$, k and l arbitrary constants

Zeros of S at $t = n\pi - l$, $n = 0, \pm 1, \pm 2, \ldots$

State Eqns.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u^*, \quad u^* = \pm 1 \end{aligned}$$

$u^* = 1$ $\quad$ $\frac{d}{dt}(x_1 - 1) = x_2$, $\dot{x}_2 = -(x_1 - 1)$ $\quad$, $\left.\begin{aligned} \xi &= x_1 - 1 \\ \dot{\xi} &= x_2 \\ \dot{x}_2 &= -\xi \end{aligned}\right\} \ddot{\xi} + \xi = 0.$

That is

$$\begin{aligned} \xi &= a\cos(t + \alpha) \\ x_2 &= -a\sin(t + \alpha) \\ x_1 - 1 &= a\cos(t + \alpha) \\ x_2 &= -a\sin(t + \alpha) \\ (x_1 - 1)^2 + x_2 &= a^2 \end{aligned}$$

$$\dot{x}_1 = -x_1 + 1$$

$$\dot{x}_2 < 0 \text{ if } x_1 > 1$$

$$u^* = -1$$

$$\frac{d}{dt}(x_1 + 1) = x_2 \qquad \xi = x_1 + 1$$

$$\dot{x}_2 = -(x_1 + 1) \qquad \ddot{\xi} + \xi = 0$$

$$x_1 + 1 = b\cos(t + \beta)$$

$$x_2 = -b\sin(t + \beta)$$

$$(x_1 + 1)^2 + x_2^2 = b^2$$

Optimal paths consist of circles, $\mathcal{C}^+$ for $u^* = 1$, $\mathcal{C}^-$ for $u^* = -1$. Optimal path to zero consists of alternate arcs of $\mathcal{C}^+$ and $\mathcal{C}^-$ curves.

$$S = \psi_2 = k \sin(t + l)$$

Switches at $t = n\pi - l$, n integer.

So the switches will be π apart in time. **In this time** $\mathcal{C}^+$ or $\mathcal{C}^-$ sweeps out a semicircle (because

$$x_1 + 1 = b \cos(t + \beta)$$
$$x_2 = -b \sin(t + \beta)$$

which is the equation of a circle parametrized by t $0 \leq t \leq 2\pi$).

The origin is actually reached on either $\mathcal{C}_1^-$ or $\mathcal{C}_1^+$

If we are on $\mathcal{C}_1^-$ (or $\mathcal{C}_1^+$) the optimal strategy is obviously to stay on it. Any other initial state must either reach $\mathcal{C}_1^+$ on a $\mathcal{C}^-$ path, or reach $\mathcal{C}_1^-$ on a $\mathcal{C}^+$ path.

Suppose we have a $\mathcal{C}^-$ path intersects $\mathcal{C}_1^+$ at Q at time τ. It must have switched to $\mathcal{C}^-$ path from a $\mathcal{C}^+$ path

At R at the time $\tau - \pi$, a semicircle away. Similarly an optimal path switching onto $\mathcal{C}^-$ must have had the previous switch on the lower half of the semicircle radius 1, centre (3, 0). Continue to work backwards in this way, R must have come from a switch $\mathcal{C}^-$ to $\mathcal{C}^+$ at R', on a semicircle of radius 1, centred at (5, 0). And so on:

$$u^* = \begin{cases} -1 & \text{above } \mathcal{C} \text{ and on } \mathcal{C}_1^- \\ +1 & \text{below } \mathcal{C} \text{ and on } \mathcal{C}_1^+ \end{cases}$$

Eigenvalues with negative real part:

Example 2. Suppose that the uncontrolled system has a stable focus at $\underset{\sim}{0}$.

$$\binom{\dot{x}_1}{\dot{x}_2} = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix} \underset{\sim}{x} + \binom{k}{1} u\,, \;\; k > 0$$

$$|u| \leq 1$$

Solution.

$$\begin{aligned} H &= -1 + \psi_1(-kx_1 + x_2 + ku) \\ &\quad + \psi_2(-x_1 - kx_2 + u) \\ u^* = \pm 1 &= sgn(k\psi_1 + \psi_2). \end{aligned}$$

Matrix A has eigenvalues $\lambda = -k \pm i$

$$u^* = 1 \qquad \left.\begin{aligned} \dot{x}_1 &= -kx_1 + x_2 + ku \\ \dot{x}_2 &= -x_1 - kx_2 + u \end{aligned}\right\} \quad \begin{aligned} &\text{critical point} \\ &(1,0) = N \end{aligned}$$

$$x_1 - 1 = ae^{-kt}\cos(t+\alpha)$$

$$x_2 = -ae^{kt}\sin(t+\alpha)$$

$u^* = -1$ Critical point $(-1, 0) = M$

$$x_1 + 1 = ae^{-kt}\cos(t+\alpha)$$

$$x_2 = -ae^{-kt}\sin(t+\alpha)$$