General optimal control problem (revision 2)

equation $\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u})$ $\quad t_0 \le t \le t_1$

$\vec{x}(t_0) = \vec{x}^0$

$\vec{x}(t_1) = \vec{x}^1$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

A cost functional

$$J = \int_{t_0}^{t_1} f_0(\vec{x}, \vec{u})\, dt$$

$\vec{u} = u$

and a set of admissible control $\vec{u}$

(piecewise continuous and restrict to a bounded domain U)

Find $u = u^*$ with corresponding path $\vec{x}^*(t)$

which transfer the system state $\vec{x}^0$ at $t = t_0$ to $\vec{x}^1$ at $t = t_1$

in such a way that $J = \int_{t_0}^{t_1} f_0(\vec{x}, \vec{u})\, dt$ is minimized.

$n = 1$ or $n = 2$

i.e. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$

$\vec{u} = u.$ one control variable

Theorem (Pontryagin maximum principle)

Let $u^*(t)$ be an admissible optimal control with optimal path $\vec{x}^* = (x_1^*(t), x_2^*(t))$ that transfer the system from $\vec{x}^0$ at $t = t_0$ to $\vec{x}^1$ at some unspecified t_1.

Let $(u^*, \vec{x}^*)$ be optimal, i.e. that minimizes

$$J = \int_{t_0}^{t_1} f_0(x_1, x_2, u)\, dt$$

$n = 2$

Set

$$H(\vec{\psi}, \vec{x}, u) = -f_0(\vec{x}, u) + \psi_1 f_1(\vec{x}, u) + \psi_2 f_2(\vec{x}, u)$$

where $\vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ are costate satisfying

$$\frac{d\psi_1}{dt} = -\frac{\partial H}{\partial x_1}, \quad \frac{d\psi_2}{dt} = -\frac{\partial H}{\partial x_2}$$

Then

(i) $$H(\vec{\psi}^*, \vec{x}^*, u^*) = \max_u H(\psi^*, \vec{x}^*, u)$$

$t_0 \le t \le t_1^*$

(ii) $$H(\psi^*(t), \vec{x}^*(t), u^*(t)) = 0, \quad t_0 \le t \le t_1^*$$

Time of Arrival Fixed

$n = 2$

$$\dot{x}_1 = f_1(x_1, x_2, u)$$

$$\dot{x}_2 = f_2(x_1, x_2, u)$$

to be controlled from $\underset{\sim}{x}^0$ at $t = t_0$ to $\underset{\sim}{x}^1$ at $t = t_1$, where t_1 is *fixed* and *known*, so as to minimize

$$J = \int_{t_0}^{t_1} f_0(x_1, x_2, u)dt.$$

Find the optimal control.

In previous examples, we needed "$H = 0$" + end-point conditions to determine arbitrary constants (from solving DE) and t_1. But here we know t_1 already, and

$$H \equiv c$$

(4)

so the endpoint conditions are sufficient to solve the problem. In fact $H \equiv C$, a constant and $C \neq 0$ is a possibility.

Example 4. $\dot{x}_1 = -x_1 + u$ to be controlled from $x_1 = 0$ at $t = 0$ to $x_1 = 2$ at $t = 1$, minimizing

$$J = \frac{1}{2}\int_0^1 (3x_1^2 + u^2)dt$$

(no constraint on $u(t)$). Find the optimal control.

Solution. Observe that t_1 is known.

Then

$$\begin{aligned} H &= -f_0 + \psi_1 f_1 \\ &= -\frac{1}{2}(3x_1^2 + u^2) + \psi_1(-x_1 + u). \end{aligned}$$

No constraint on u, we maximinize H by considering

$$\begin{aligned} 0 = \partial H/\partial u &= -u + \psi_1 \Rightarrow u = \psi_1 \\ \partial^2 H/\partial u^2 &= -1 < 0, \text{ so a maximum.} \end{aligned}$$

(5)

Costate equations

$$\dot{\psi}_1 = -\frac{\partial H}{\partial x_1} = 3x_1 + \psi_1$$

(Optimal) State equation $\dot{x}_1 = -x_1 + \psi_1$

$\ddot{x}_1 = 4x_1$

$$\begin{aligned} \ddot{x}_1 &= -\dot{x}_1 + \dot{\psi}_1 = x_1 - \psi_1 + 3x_1 + \psi_1 \\ &= 4x_1 \end{aligned}$$

$\lambda^2 = 4 \Rightarrow \lambda = \pm 2$

$$\begin{aligned} \Rightarrow \quad x_1 &= Ae^{2t} + Be^{-2t} \\ \psi_1 &= \dot{x}_1 + x_1 \\ &= 2Ae^{2t} - 2Be^{-2t} + Ae^{2t} + Be^{-2t} \\ &= 3Ae^{2t} - Be^{-2t} \\ &= u^*. \end{aligned}$$

End conditons:

$$\begin{aligned} x_1 &= 0 \qquad \text{at } t = 0 \\ x_1 &= 2 \qquad \text{at } t = 1 \end{aligned}$$

(6)

$$
\begin{aligned}
0 &= A + B \qquad \Rightarrow \qquad B = -A \\
2 &= Ae^2 + Be^{-2} = A(e^2 - e^{-2}) = 2A \sinh 2 \\
A &= 1/\sinh 2.
\end{aligned}
$$

Optimal control is

$$
u^* = \frac{1}{\sinh 2}(3e^{2t} + e^{-2t})
$$

(Considerably simpler because we knew t_1).

Optimal time control

~~# Case 3, imaginary and complex eigenvalues later if time permits~~

$n = 2$

Return to $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{l}u$, $\quad f_0 = 1$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$, $\vec{l} = \begin{pmatrix} l \\ m \end{pmatrix}$

(1) $\quad |u| \leq 1$

$$
\begin{aligned}
H &= \sum \psi_i f_i \\
&= -1 + \psi_1(ax_1 + bx_2 + lu) \\
&\qquad + \psi_2(cx_1 + dx_2 + mu) \\
&= -1 + \psi_1(ax_1 + bx_2) + \psi_2(cx_1 + dx_2) \\
&\qquad + u(l\psi_1 + m\psi_2)
\end{aligned}
$$

optimal time.

$S = (l\psi_1 + m\psi_2)$

$$
\begin{aligned}
\dot{\psi}_1 &= -a\psi_1 - c\psi_2 \\
\dot{\psi}_2 &= -b\psi_1 - d\psi_2
\end{aligned}
$$

$$
\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \dot{\underset{\sim}{\psi}} = -A^T\vec{\psi}.
$$

(2) MAXIMIZE H (as a function of u)

Linear in u, so $u^* = \pm 1$, depending on the sign of $l\psi_1 + m\psi_2$:

$$
u^* = sgn(l\psi_1 + m\psi_2). = \begin{cases} 1, & l\psi_1 + m\psi_2 > 0 \\ -1 & l\psi_1 + m\psi_2 < 0 \end{cases}
$$

(8)

Piecewise continuous controls, switch when $l\psi_1 + m\psi_2$ changes sign (i.e. at zeros of $l\psi_1 + m\psi_2$)

$$S = l\psi_1(t) + m\psi_2(t)$$

SWITCHING FUNCTION.

Between any two adjacent zeros of S, u^* is constant:

State Equation $\quad \underset{\sim}{\dot{x}} = A\underset{\sim}{x} + \underset{\sim}{l}u^*, \quad u^* = +1 \text{ or } -1.$

If $\det A \neq 0$, trajectories for $u^* = 1$ have an isolated singularity (equilibrium points) at the solution of

$$\begin{aligned} ax_1 + bx_2 + l &= 0 \\ cx_1 + dx_2 + m &= 0. \end{aligned}$$

$A\vec{x} = -\binom{l}{m}$

For $u^* = -1$, the equilibrium is at the solution of

$$\begin{aligned} ax_1 + bx_2 - l &= 0 \\ cx_1 + dx_2 - m &= 0. \end{aligned}$$

Behaviour of both families of trajectories is governed by the eigenvalues of the "system matrix" A. The pattern is the same as that of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, translated to the equilibrium point.

If A has real eigenvalues, $-A^T$ has real eigenvalues also:

$$\begin{aligned} A: &\quad \lambda^2 - (a+d)\lambda + \det A = 0 \\ A^T: &\quad \lambda^2 + (a+d)\lambda + \det A = 0. \end{aligned}$$

Solution of costate equations has the form

$$\underset{\sim}{\psi} = \underset{\sim}{h} e^{q_1 t} + \underset{\sim}{k} e^{q_2 t}$$

q_1, q_2 eigenvalues of $-A^T$

$\underset{\sim}{h}$, $\underset{\sim}{k}$ corresponding eigenvectors.

Hence, switching function is of the form

$$\begin{aligned} S &= l\psi_1 + m\psi_2 \\ &= Le^{q_1 t} + Me^{q_2 t} \end{aligned}$$

has at most one zero.

Lemma. (i) If eigenvalues of A are real, the switching function has at most one zero.

(ii) Only possible optimal control sequences are

$u^* = 1, \quad t_0 \leq t \leq t_1$

$u^* = -1, \quad t_0 \leq t \leq t_1$

$u^* = \begin{cases} +1, & t_0 \leq t \leq \tau \\ -1, & \tau \leq t \leq t_1 \end{cases}$ τ is a zero of S.

$u^* = \begin{cases} -1, & t_0 \leq t \leq \tau \\ +1, & \tau \leq t \leq t_1 \end{cases}$

Remark. It can be shown that PMP is both necessary and sufficient for linear time-optimal control problems. Hence, if we can find a control sequence of the type on the lemma, it must be an optimal control.

~~**Example.**~~ Question.

$$\dot{x}_1 = -3x_1 + 2x_2 + 5u$$
$$\dot{x}_2 = 2x_1 - 3x_2$$

to control from any initial state to the origin in minimum time, $|u| \leq 1$. Find the optimal control u^* mini-

Example 2.

11

$$\dot{x}_1 = 3x_1 + 2x_2 + 5u$$
$$\dot{x}_2 = 2x_1 + 3x_2, \qquad |u| \le 1$$

Control to $\underset{\sim}{0}$ in minimum time.

16

Solution.

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \underset{\sim}{x} + u \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

(1)

1, 5

$$H = -1 + \psi_1(3x_1 + 2x_2 + 5u) + \psi_2(2x_1 + 3x_2)$$
$$= -1 + \psi_1(3x_1 + 2x_2) + \psi_2(2x_1 + 3x_2) + u(5\psi_1)$$
$$\dot{\underset{\sim}{\psi}} = -A^T \underset{\sim}{\psi} = \begin{pmatrix} -3 & -2 \\ -2 & -3 \end{pmatrix} \underset{\sim}{\psi}.$$

(2) H is linear in u, $|u| \le 1$

$\Rightarrow H$ max at $u^* = sgn(5\psi_1) = \pm 1$

Optimal trajectories satisfy

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \underset{\sim}{x} + u^* \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad u^* = \pm 1$$

Eigenvalues of A are 1, 5 UNSTABLE NODE

(3)

Eqlbrm point for $u^* = 1$

$$\left.\begin{aligned} 3x_1 + 2x_2 + 5 &= 0 \\ 2x_1 + 3x_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -3 \\ x_2 &= +2 \end{aligned}$$

Eqn for $u^* = -1$

$$\left.\begin{aligned} 3x_1 + 2x_2 - 5 &= 0 \\ 2x_1 + 3x_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 3 \\ x_2 &= -2 \end{aligned}$$

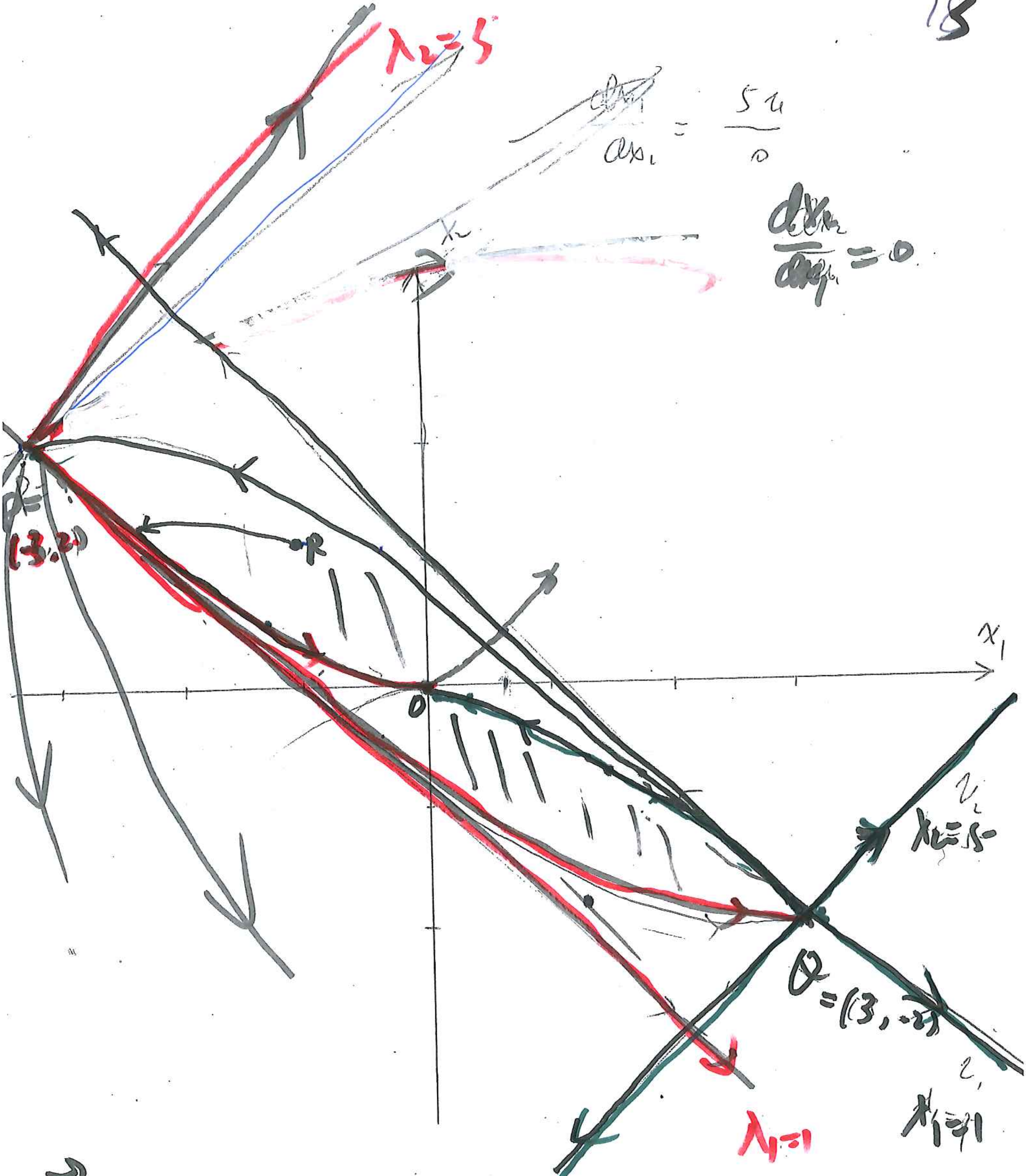
The general solution for $\dot{\vec{x}} = A\vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $u^* = 1$

$$\vec{x}(t) = \alpha \vec{v}_1 e^{t} + \beta \vec{v}_2 e^{5t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

* critical pour for $u^* = -1$ is $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$$\vec{x}(t) = \alpha \vec{v}_1 e^{t} + \beta \vec{v}_2 e^{5t} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$$\vec{X}(t) = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \begin{matrix} P \\ Q \end{matrix}$$

$u^* = \begin{cases} -1 & \text{above POQ or OQ} \\ +1 & \text{below POQ} \end{cases}$ inside region

Lecture 27 <14>

Larger K, $|u| \le K$, larger region of controllability.

Example 3.

$$\begin{aligned} \dot{x}_1 &= x_1 + 3x_2 - 7u \\ \dot{x}_2 &= 3x_1 + x_2 - 5u \end{aligned}, \quad |u| \le 1$$

control to origin in minimum time.

$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

Solution.

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} -7 \\ -5 \end{pmatrix} u$$

, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(1)

$$\begin{aligned} H = -1 + \psi_1(x_1 + 3x_2) + \psi_2(3x_1 + x_2) \\ +u(-7\psi_1 - 5\psi_2) \end{aligned}$$

$S = -7\psi_1 - 5\psi_2$

(2) Maximized (for $|u| \le 1$) if

$$u^* = sgn(-7\psi_1 - 5\psi_2) = \pm 1.$$

Optimal trajectories satisfy

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} x + u^* \begin{pmatrix} -7 \\ -5 \end{pmatrix}, \quad u^* = \pm 1$$

(3) Eigenvalues of A:

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = -2,\ 4 \text{ saddle point (unstable)}$$

Eigenvectors: $\lambda_1 = -2$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 \\ -2v_2 \end{pmatrix}$$

$\lambda_1 = -2 \qquad v_1 + 3v_2 = -2v_1,\ v_2 = -v_1;\ \binom{1}{-1} = \vec{v}_1$

Eigenvectors: $\lambda_2 = 4$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix}$$

$\lambda_2 = 4 \qquad v_1 + 3v_2 = 4v_1,\ v_2 = v_1;\ \binom{1}{1} = \vec{v}_2$

$\vec{x}(t) = \alpha \vec{v}_1 e^{\lambda_1 t} + \beta \vec{v}_2 e^{\lambda_2 t} + \begin{matrix} P \\ Q \end{matrix}$

46

(4) **Eqlbrm** at $u^* = 1$,

$$\begin{cases} x_1 + 3x_2 - 7 = 0 \\ 3x_1 + x_2 - 5 = 0 \end{cases}$$

$$3x_1 + 9x_2 - 21 = 0$$

$$8x_2 = 16, \quad \left.\begin{matrix} x_2 = 2 \\ x_1 = 1 \end{matrix}\right\} \quad P = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Eqlbrm at $u^* = -1$

$$x_1 + 3x_2 + 7 = 0$$

$$\left.\begin{matrix} 3x_1 + 2x_2 + 5 = 0 \\ 3x_1 + 9x_2 + 21 = 0 \end{matrix}\right\} \quad \left.\begin{matrix} x_2 = -2 \\ x_1 = -1 \end{matrix}\right\} \quad Q = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

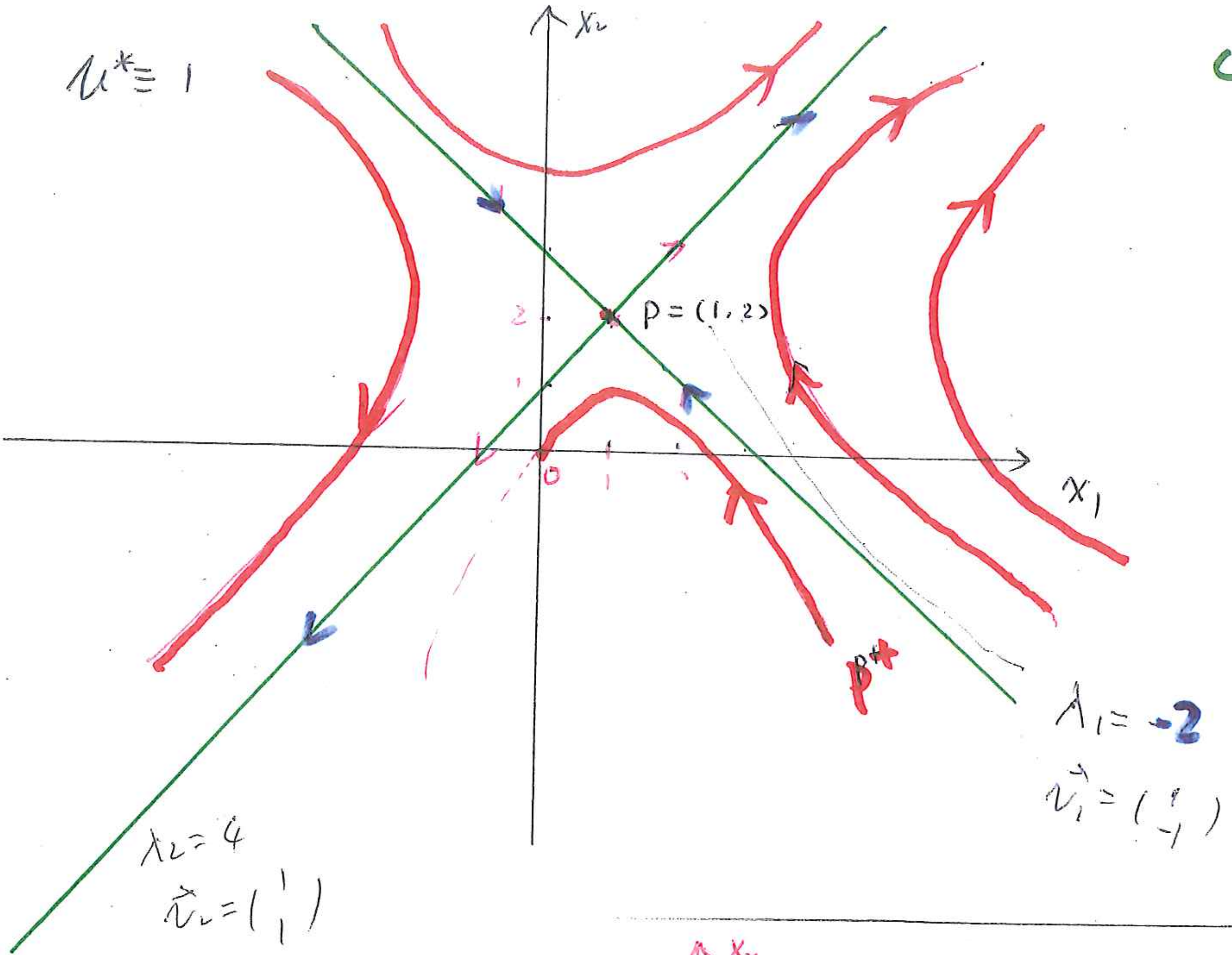
$$x(t) = \alpha \vec{v}_1 e^{\lambda_1 t} + \beta \vec{v}_2 e^{\lambda_2 t} + \begin{matrix} P \\ Q \end{matrix}$$

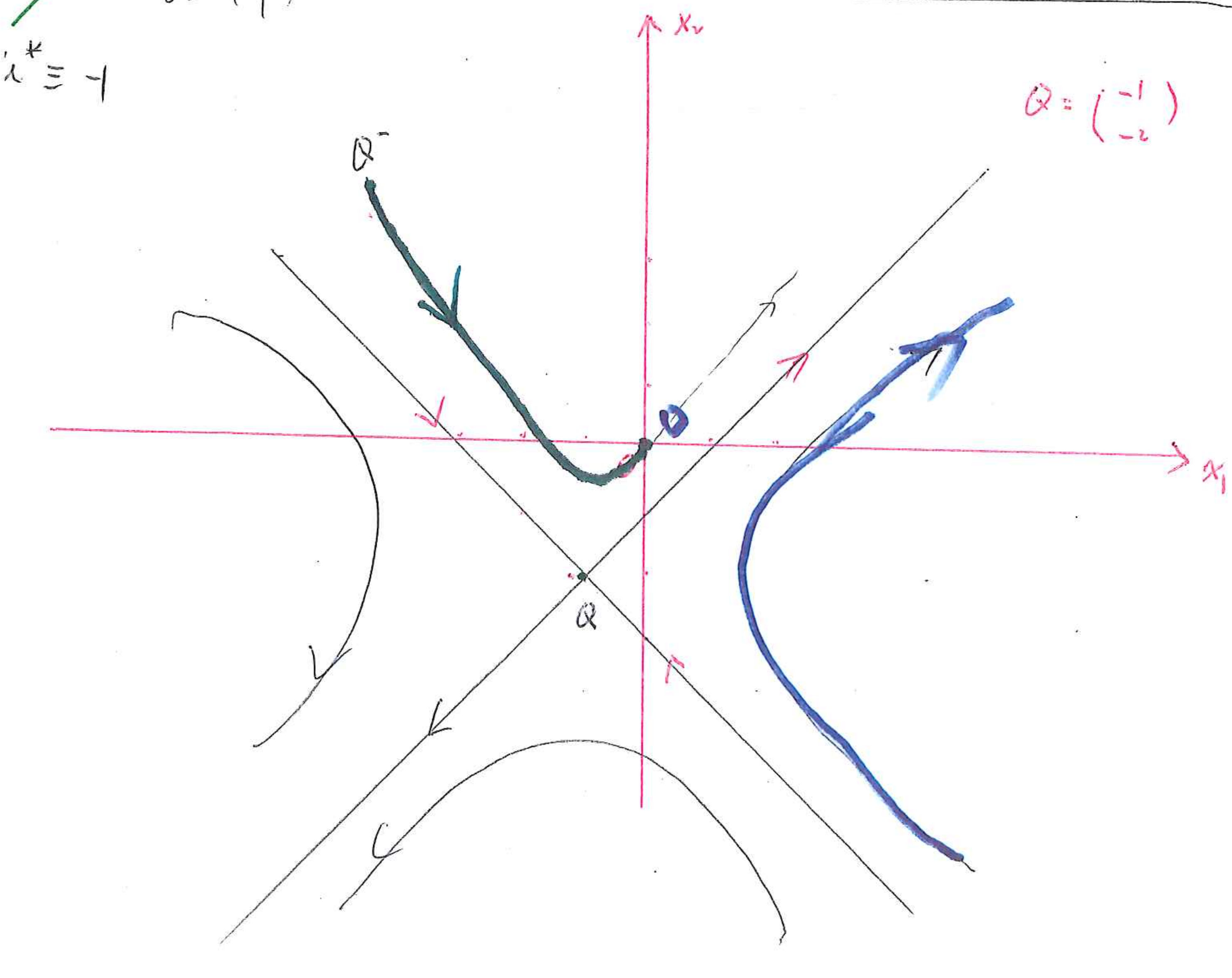
$u^* = 1$

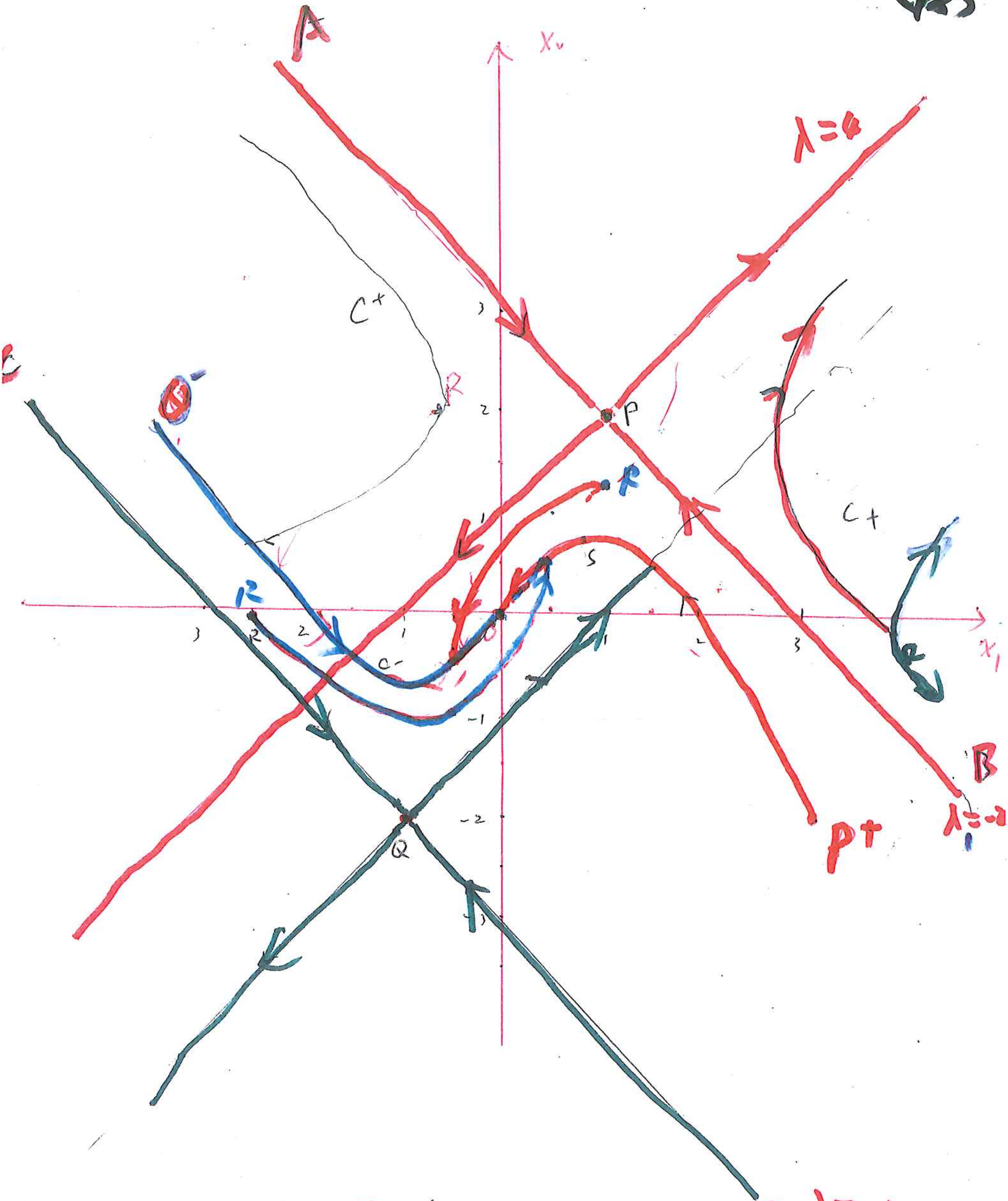
C_+ $$x(t) = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$u^* = -1$

C_- $$x(t) = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$


$u^* \equiv 1$
x_2
x_1
$p = (1, 2)$
p^+
$\lambda_1 = -2$
$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\lambda_2 = 4$
$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$


$u^* \equiv -1$
x_2
x_1
$Q = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$
Q^-
Q



$$u^* = \begin{cases} -1 & \text{below } O^-OP^+ \text{ \& on } O^-O \\ +1 & \text{above } O^-OP^+ \text{ \& on } OP^+ \end{cases}$$

Real part is non-negative

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} k \\ 1 \end{pmatrix} u, \quad k > 0$$

$$|u| \leq 1$$

$A = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix}$ has eigenvalues $\lambda = -k \pm \sqrt{-1}$

PMP

$$H = -1 + \psi_1(-k x_1 + x_2) + \psi_2(k\psi_1 + \psi_2)$$
$$+ u(k\psi_1 + \psi_2)$$

(*) $S = k\psi_1 + \psi_2$

(*) $u^* = \begin{cases} 1 & \text{if } S > 0 \\ -1 & \text{if } S < 0 \end{cases}$

H is max at $u = u^*$

The corresponding orbits for $u^*=1$

$$\begin{cases} x_1 = 1 + a e^{-kt} \cos(t+\alpha) \\ x_2 = -a e^{-kt} \sin(t+\alpha) \end{cases}$$

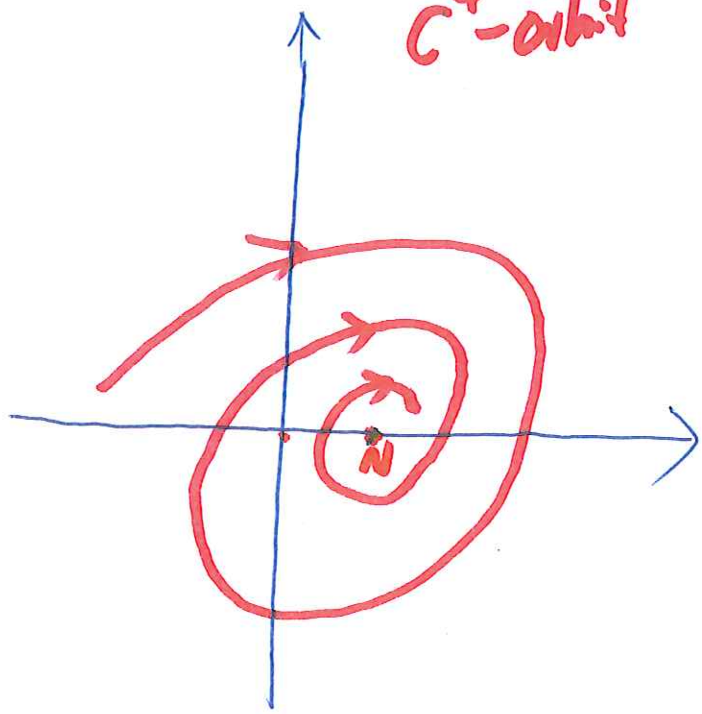
a, α are constants. $a>0$

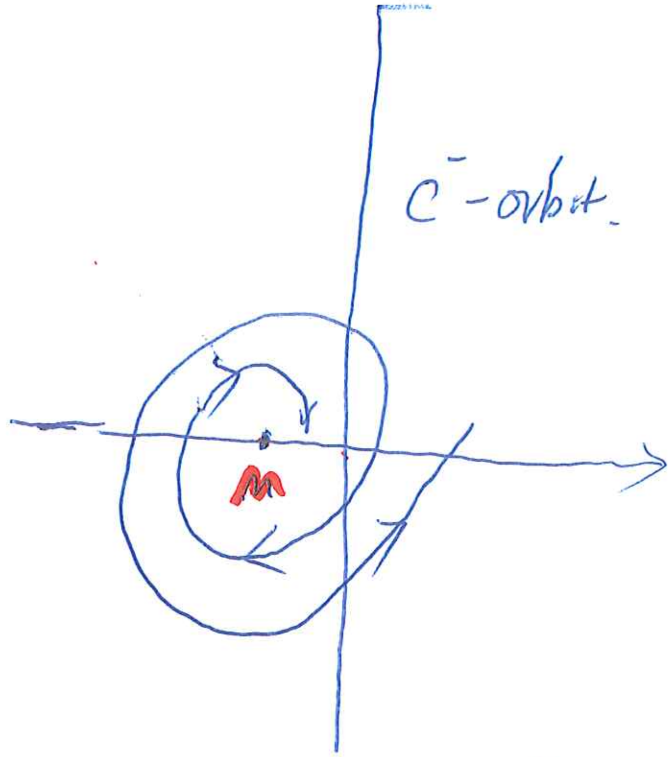
$N=(1,0)$ is singularity

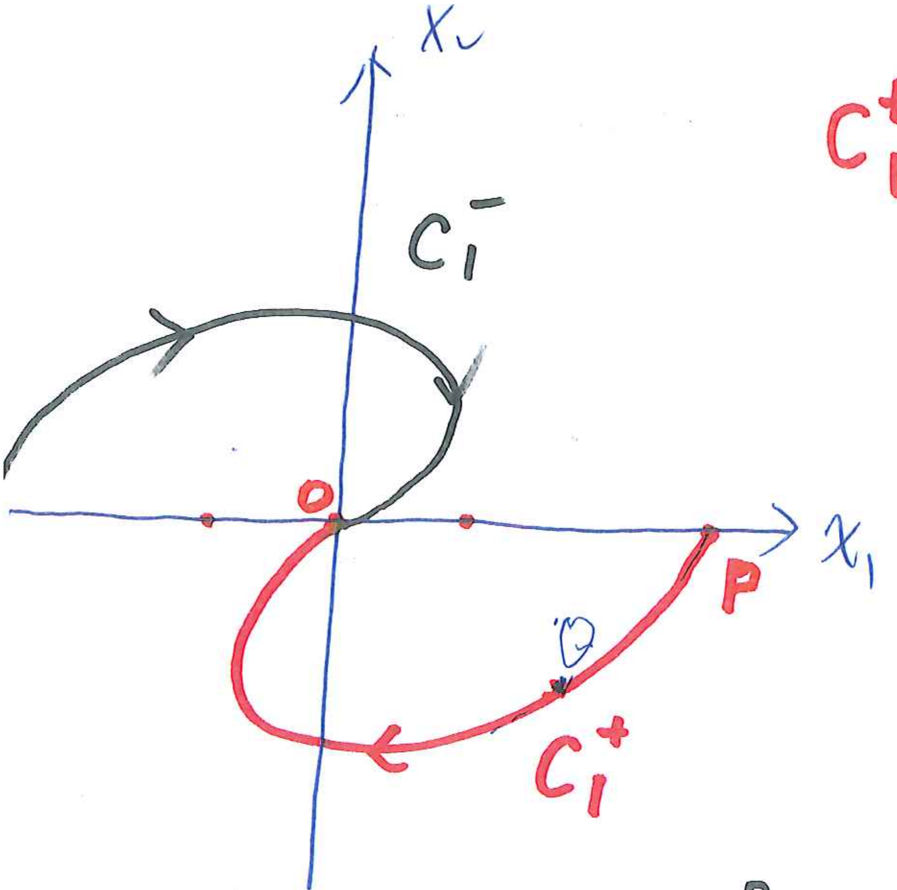
for $u^*=-1$,

$$\begin{cases} x_1 = -1 + a e^{-kt} \cos(t+\alpha) \\ x_2 = -a e^{-kt} \sin(t+\alpha) \end{cases}$$

$M=(-1,0)$ is singularity







$$C_1^+ : \begin{cases} x_1 = 1 - e^{-k\sigma} \cos\sigma \\ x_2 = e^{-k\sigma} \sin\sigma \end{cases}$$

$\forall \sigma \in [-\pi, 0]$

$$C_1^- = \begin{cases} x_1 = -1 + e^{-k\sigma} \cos\sigma \\ x_2 = e^{-k\sigma} \sin\sigma \end{cases}$$

$\sigma \in [-\pi, 0]$

The co-state equation

$$\begin{cases} \dot{\psi}_1 = -\dfrac{\partial H}{\partial x_1} = k\psi_1 + \psi_2 \\ \dot{\psi}_2 = -\dfrac{\partial H}{\partial x_2} = -\psi_1 + k\psi_2 \end{cases}$$

has soln

$$\psi_1 = b e^{kt} \cos(t+\beta)$$

$$\psi_2 = -b e^{kt} \sin(t+\beta)$$

$$S = k\psi_1 + \psi_2 = kbe^{kt}\cos(t+\beta) - be^{kt}\sin(t+\beta)$$

$$= c e^{kt} \sin(t+\nu)$$

has zero at

$$t = -\nu \pm n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

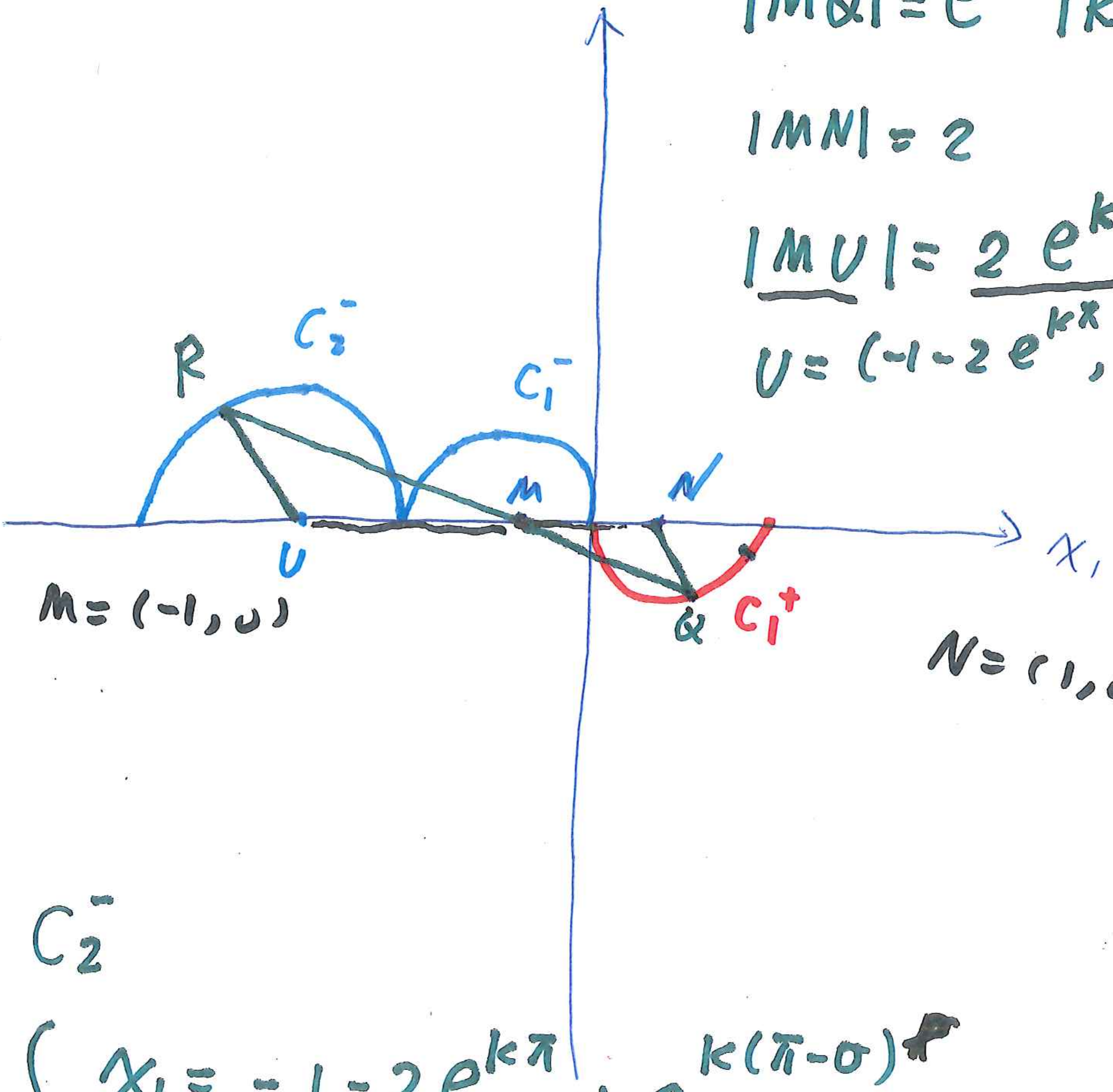
$\Delta RMU \sim \Delta MNQ$

$|MQ| = e^{-k\pi} |RM|$

$|MN| = 2$

$|MU| = 2e^{k\pi}$

$U = (-1-2e^{k\pi}, 0)$



$M = (-1, 0)$

$N = (1, 0)$

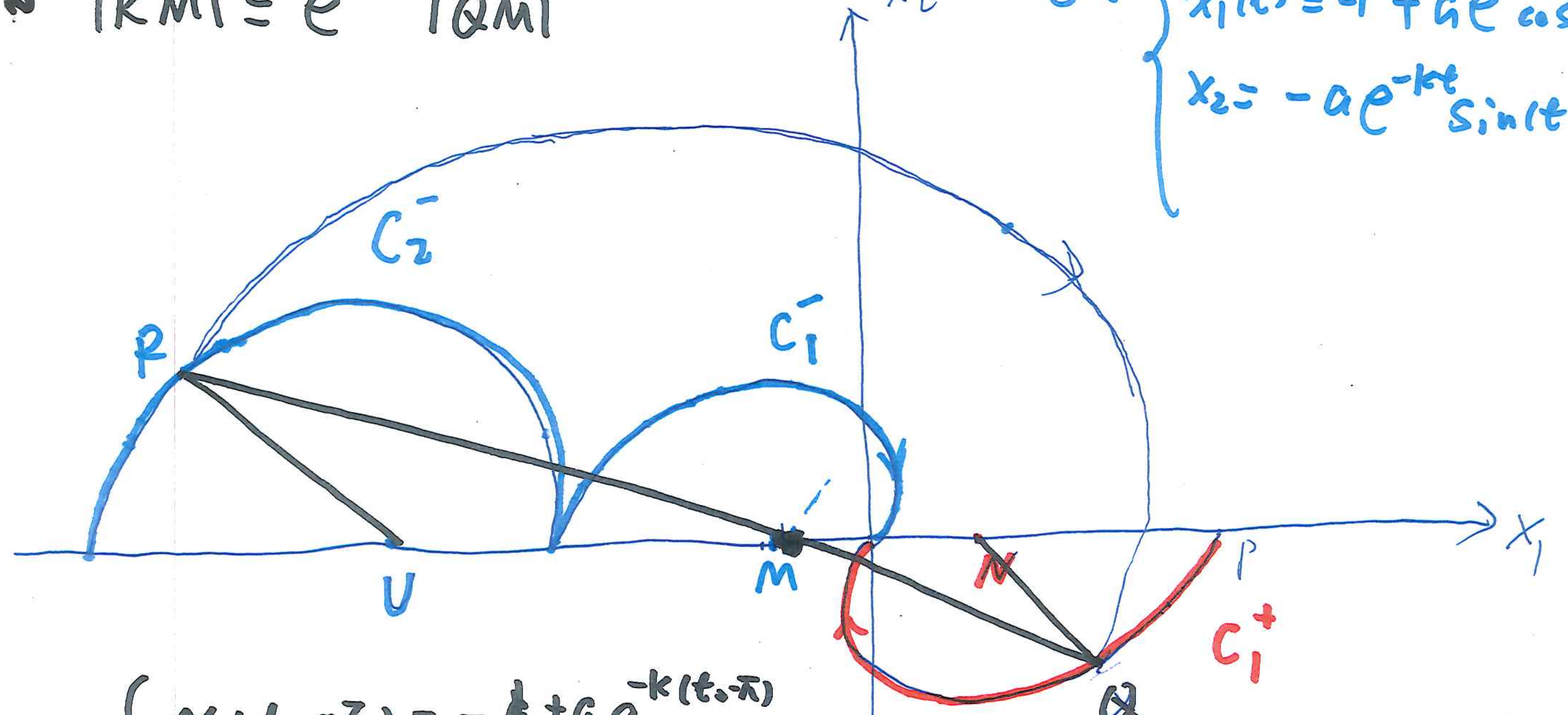
C_2^-

$$\begin{cases} x_1 = -1 - 2e^{k\pi} + e^{k(\pi-\sigma)} \cos\sigma \\ x_2 = -e^{k(\pi-\sigma)} \sin\sigma \end{cases}$$

$\sigma \in [-\pi, 0]$

2. $|RM| = e^{\pi k} |QM|$

$C^- : \begin{cases} x_1(t) = -1 + a e^{-kt} \cos(t+\alpha) \\ x_2 = -a e^{-kt} \sin(t+\alpha) \end{cases}$

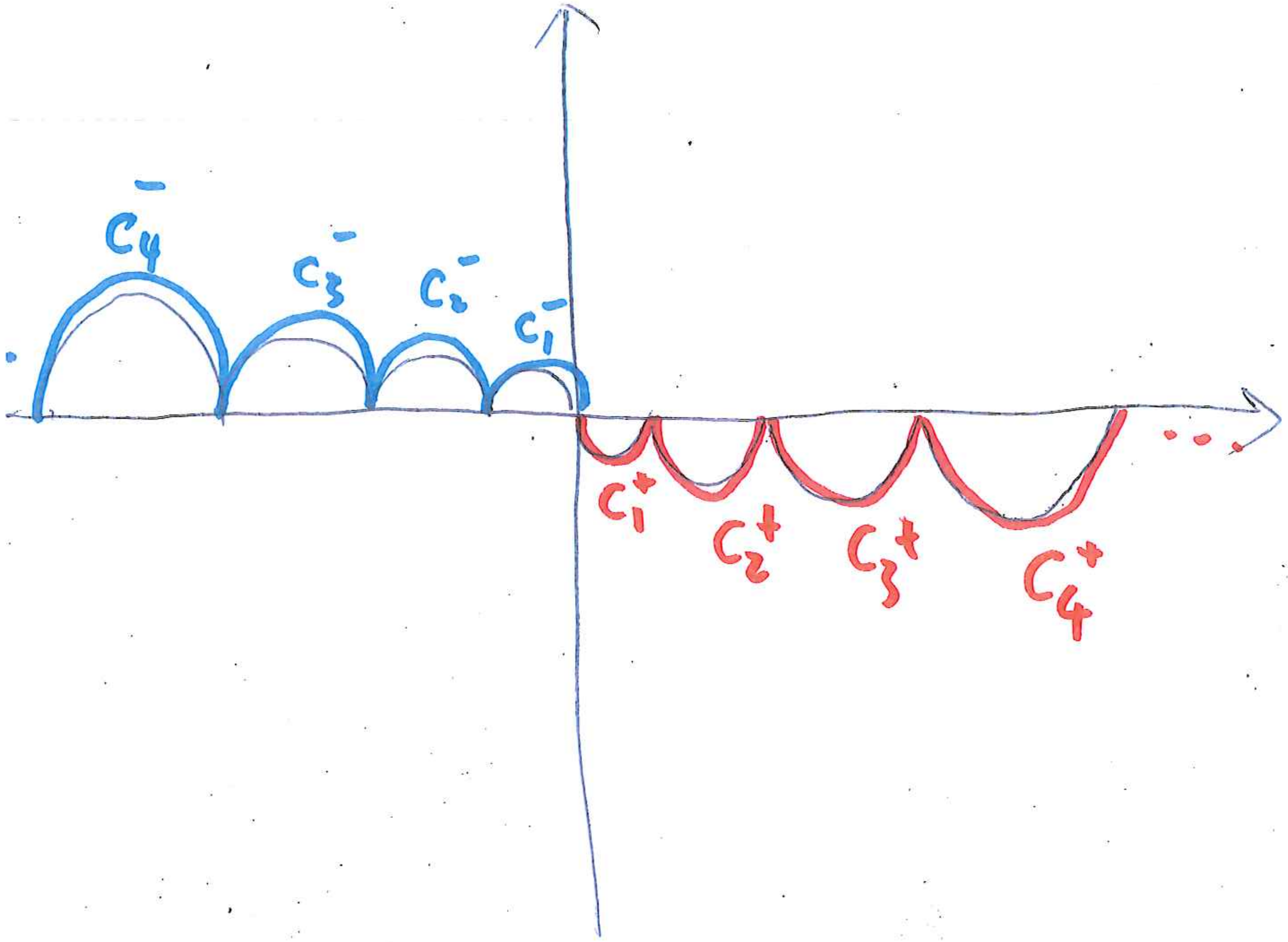


$R = \begin{cases} x_1(t_0 - \pi) = -1 + a e^{-k(t_0-\pi)} \cos(t_0 - \pi + \alpha) \\ x_2(t_0 - \pi) = -a e^{-k(t_0-\pi)} \sin(t_0 - \pi + \alpha) \end{cases}$

$Q = \begin{cases} x_1(t_0) = -1 + a e^{-kt_0} \cos(t_0+\alpha) \\ x_2(t_0) = -a e^{-kt_0} \sin(t_0+\alpha) \end{cases}$

$|RM| = |a| e^{\pi k} e^{-kt_0}$

$|QM| = |a| e^{-k t_0}$



The switching curve Γ. The loop grow in size

$$u^* = \begin{cases} -1 & \text{above } \Gamma \text{ and on } C_1^- \\ +1 & \text{below } \Gamma \text{ and } C_1^+ \end{cases}$$

26

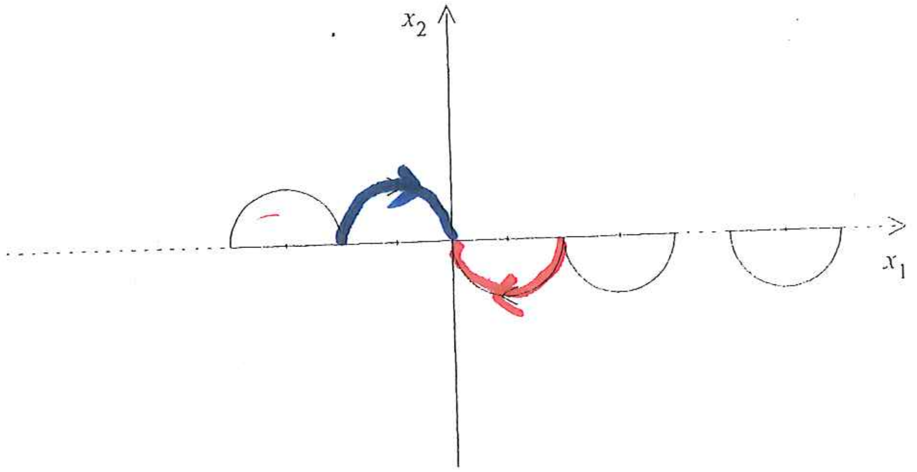


Figure 5.17 The switching curve is a series of semicircular loops

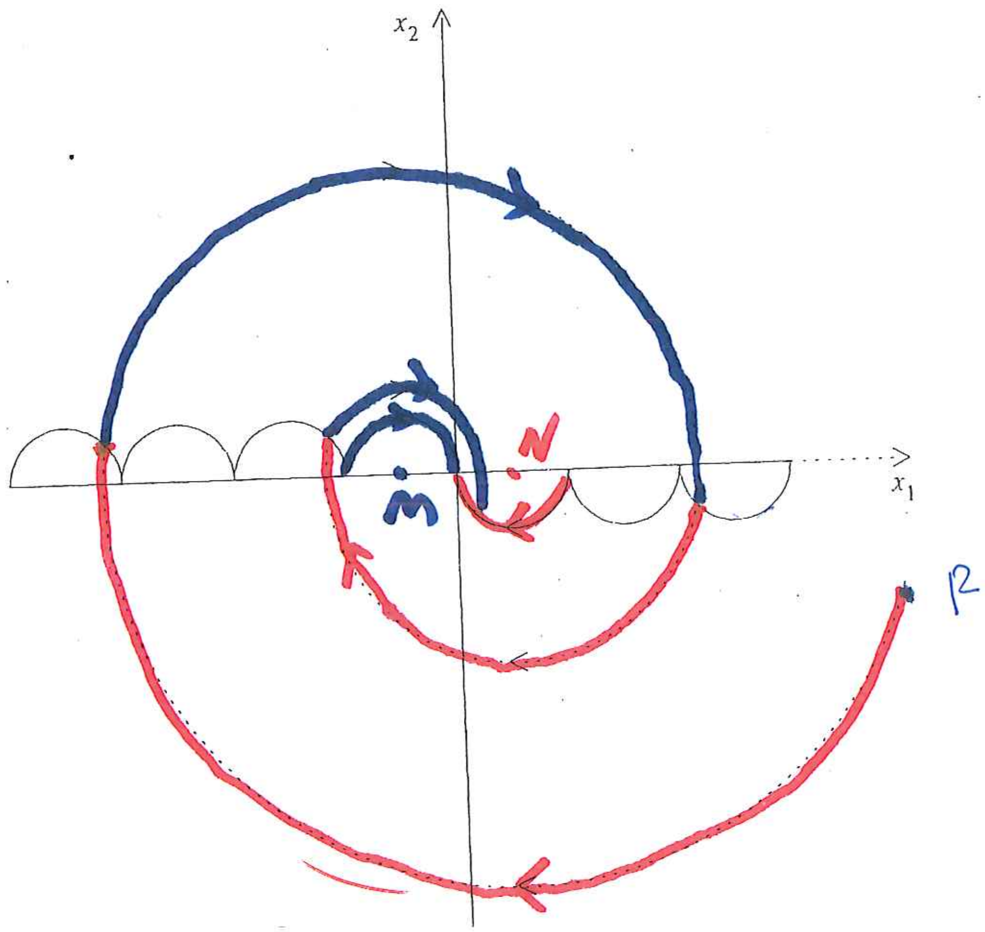


Figure 5.18 The switching curve and typical optimal paths

will, when slightly disturbed from equilibrium, perform small oscillations that are solutions of this equation.

- Transversality conditions (C of V)
- Complex eigenvalues switching curve
- Riccati equation

Apply Pontryagin Max Princ:

$$\begin{aligned} H &= \psi_0 \dot{X}_0 + \psi_1 f_1 + \cdots + \psi_n f_n \\ &= \psi_0 \left\{ f_0 + \sum_1^n \frac{\partial g}{\partial x_j} f_j \right\} + \sum_1^n \psi_j f_j . \end{aligned}$$

As previously, take $\psi_0 = -1$ and the costate equations are

$$\dot{\psi}_i = -\partial H / \partial x_i \ , \quad i = 1, ..., n.$$

However, these are more complicated than the case where J does not involve $x(t_1)$. Let's look for a simplification.

Rearrange H

$$H' = -f_0 + \sum_1^n \left[\psi_j - \partial g / \partial x_j \right] f_j .$$

Introduce "pseudo-costate" variables

$$\lambda_i = \psi_i - \partial g/\partial x_i \,, \quad i = 1, 2, ..., n$$

$$H = -f_0 + \sum_1^n \lambda_j f_j := H'.$$

It turns out that

$$\dot{\lambda}_i = -\frac{\partial H'}{\partial x_i}\,, \quad i = 1, ..., n$$

* The λ_i's formally act like costate variables and the equations are much easier to solve.

x(t1) is free

\# Since $\underset{\sim}{x}(t_1) = \underset{\sim}{x}^1$ free, so the transversality condition is

$$\Longrightarrow \begin{cases} \psi_i(t_1) = 0\,, & i = 1, ..., n. \\ \lambda_i(t_1) = -\frac{\partial g}{\partial x_i}(t_1) & i = 1, ..., n. \end{cases}$$

Summary:

- Write $H' = -f_0 + \sum_j \lambda_j f_j$
- Maximize H' as a function of u.

- End conditions $\underset{\sim}{x}(t_0) = \underset{\sim}{x}^0$

Two endpoint

Boundary Value

Problem.

$$\lambda_i(t_1) = -\frac{\partial g}{\partial x_i}\bigg|_{t=t_1}$$

Example. $\dot{x} = -\alpha x + u$, controlled from $x = 0$ at $t = 0$ to $x(t_1)$ at a fixed time t_1, minimizing

$$J = -x(t_1) + \int_0^{t_1} u^2 dt.$$

$g(x(t_1)) = -x(t_1)$

Find the optimal control u^*.

(Control u is unconstrained, but the u^2 term makes it expensive to use too much.)

Solution.

$$\begin{array}{c} \quad\ -f_0 \qquad\quad \lambda_1 f_1 \\ H = H' = -u^2 + \lambda(-\alpha x + u) \end{array}$$

Costate equations:

$$\dot{\lambda} = -\partial H'/\partial x$$

$$\dot{\lambda} = \alpha\lambda, \quad \boxed{\lambda = Ae^{\alpha t}}$$

To maximize H' as a function of u,

$$\frac{\partial H'}{\partial u} = H'_u = -2u + \lambda = 0 \qquad \boxed{u^* = \lambda/2}$$

So $u^* = Ae^{\frac{\alpha t}{2}}$ $= \frac{A}{2} e^{\alpha t}$

Optimal state equation

$$\frac{dx}{dt} = \dot{x} = -\alpha x + u^* = -\alpha x + Ae^{\alpha t}/2$$

$$\Rightarrow \quad \boxed{x = Be^{-\alpha t} + Ae^{\alpha t}/4\alpha}$$

End conditions:

$$x(0) = 0$$

$$0 = B + A/4\alpha\,, \quad B = -A/4\alpha$$

$$\text{At} \quad t = t_1\,, \quad \lambda = -\tfrac{\partial g}{\partial x}$$

$$\text{Now} \quad g(x(t_1)) = -x(t_1)\,, \qquad \text{so } g(x) = -x$$

$$\Rightarrow \quad \lambda(t_1) = -\tfrac{\partial g}{\partial x} = +1\,, \quad Ae^{\alpha t_1} = +1$$

$$A = +e^{-\alpha t_1}$$

Hence $u^* = +\frac{e^{-\alpha t_1}}{2}e^{\alpha t} = +\frac{1}{2}e^{\alpha(t-t_1)}$.

The optimal trajectory is

$$\begin{aligned} x &= -e^{-\alpha t_1}e^{-\alpha t} + \frac{1}{4\alpha}e^{\alpha(t-t_1)} \\ &= \frac{e}{2\alpha}\left[\frac{e^{\alpha(t)} - e^{-\alpha(t)}}{2}\right] \\ &= \frac{e^{-\alpha t_1}}{2\alpha}\sinh \alpha t. \end{aligned}$$

This example we have just done is close to a special case of a wide class of useful and realistic systems: