

MATH3403: Tutorial 2

Classification, transformation and solution of 2nd order linear pdes

1. 2nd order linear PDE's

Under an arbitrary transformation $\xi(x, y)$ $\eta(x, y)$ the term $a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy}$ becomes:

for-
mula/pde01.tex

$$\begin{aligned} & u_{\xi\xi} (a(x, y)(\xi_x)^2 + b(x, y)\xi_x\xi_y + c(x, y)(\xi_y)^2) \\ & + u_{\xi\eta} (2a(x, y)\xi_x\eta_x + b(x, y)(\xi_x\eta_y + \xi_y\eta_x) + 2c(x, y)\xi_y\eta_y) \\ & + u_{\eta\eta} (a(x, y)(\eta_x)^2 + b(x, y)\eta_x\eta_y + c(x, y)(\eta_y)^2) \\ & + u_{\xi} (a(x, y)\xi_{xx} + b(x, y)\xi_{xy} + c(x, y)\xi_{yy}) \\ & + u_{\eta} (a(x, y)\eta_{xx} + b(x, y)\eta_{xy} + c(x, y)\eta_{yy}) \end{aligned}$$

2. Under the transform $\xi(x, y) = \alpha x + \beta y$, $\eta(x, y) = \gamma x + \delta y$ the term $au_{xx} + bu_{xy} + cu_{yy}$ becomes:

pde/cov004.tex

$$u_{\xi\xi} (a\alpha^2 + b\alpha\beta + c\beta^2) + u_{\eta\eta} (a\gamma^2 + b\gamma\delta + c\delta^2) + u_{\xi\eta} (2a\alpha\gamma + b\alpha\delta + b\beta\gamma + 2c\beta\delta)$$

	Hyperbolic	Elliptic	Parabolic
Discriminant	$D = b^2 - 4ac > 0$	$D = b^2 - 4ac < 0$	$D = b^2 - 4ac = 0$
ξ	$\xi = (D^{\frac{1}{2}} - b)x + 2ay$	$\xi = (2cx - by)(-D)^{-\frac{1}{2}}$	$\xi = bx - 2ay$
η	$\eta = (-D^{\frac{1}{2}} - b)x + 2ay$	$\eta = y$	$\eta = y$
New form	$-4aDw_{\xi\eta}$	$c(w_{\xi\xi} + w_{\eta\eta})$	$cw_{\eta\eta}$

Derive the final row of the table for the stated transformations.

3. Classify the equation $u_{xx} + 4u_{xy} + 4u_{yy}$ and reduce it to a standard form. State the required transformation.

pde/cov008.tex

4. Find the general solution of the equation

pde/cov016.tex

$$u_{xx} + 2u_{xy} + u_{yy} = 2.$$

Verify your answer $u(x, y)$ by substitution into the equation.

5. Classify the equation $u_{xx} + 2u_{xy} + 5u_{yy}$ and reduce it to a standard form. State the required transformation.

pde/cov007.tex

6. Consider the pde

pde/cov005a.tex

$$3y^2u_{xx} - 4xyu_{xy} + x^2u_{yy} = 0.$$

Find the pde satisfied by $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ where $\xi = x^2 + 3y^2$ and $\eta = x^2 + y^2$.

7. If $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$ show that

pde/cov010.tex

$$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad \theta_x = -\frac{y}{r^2} \quad \& \quad \theta_y = \frac{x}{r^2}.$$

8. Consider the functions $u(x, y) = \tan^{-1} \frac{y}{x}$ and $u(x, y) = x^2 - y^2$. These are velocity potentials (for an inviscid fluid) of vortex flow and flow near a wall. In each case show that $u_{xx} + u_{yy} = 0$, find $w(r, \theta) = u(r \cos \theta, r \sin \theta)$, and show that w satisfies $w_{rr} + r^{-1}w_r + r^{-2}w_{\theta\theta} = 0$ (Laplace's equation in polar coordinates).

pde/cov011.tex
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9. This is a reference problem for you to attempt. It is too long for an exam or assignment question. Derive Laplace's equation in spherical coordinates using repeated application of the chain rule.

pde/cov002.tex