MATH 3402

TUTORIAL SHEET 7

1. The function $f : (A, d) \to (A, d)$ has the property that

$$d(f(x), f(y)) < d(x, y) \; \forall \; x \neq y \in A \; .$$

Show that if f has a fixed point in A then it is unique.

2. The real function f is continuous from $[a, b]$ to $[a, b]$.
Show that f has a fixed point in $[a, b]$.

3. Determine to 3dp the solution of the equation

$$x^3 = 6x + 6 \; .$$

(The solution is approximately 3.)

4. For fixed $a \in \mathbb{R}$, we define the mapping from $\mathbb{R} \setminus \{-1\}$ to $\mathbb{R}$ by

$$f(x) = 1 + \frac{a}{1+x} \; .$$

(a) For which values of a does this mapping have a fixed point?

(b) For which values of a is the mapping a (local) contraction mapping?

(c) Starting with $w_0 = 1$, for which values of a does the sequence generated by

$$w_{n+1} = f(w_n)$$

converge?

If we choose $a \in \mathbb{C}$ instead, show that the function is a contraction mapping provided $Re(\sqrt{1+a}) \geq \epsilon > 0$.

Hint: $a = (\sqrt{1+a} - 1)(\sqrt{1+a} + 1)$.

5. Let $C(0, \frac{1}{2})$ be the set of real functions continuous on $[0, \frac{1}{2}]$, together with the uniform sup metric.

Define $f : C(0, \frac{1}{2}) \to C(0, \frac{1}{2})$ by

$$(f(x))(t) = t(x(t) + 1) \; .$$

Show that f is a contraction mapping, and determine its fixed point.

Verify that the sequence starting with $x_0(t) = t$ converges to the fixed point.