MATH 3402

TUTORIAL SHEET 6

1. Show that $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

2. Verify that the collection $\mathcal{T}_Y$ defined for the one-point compactification process is a topology on Y.

3. Let $Y = \mathbb{R} \cup \{y\}$ be the one-point compactification of $\mathbb{R}$ with the usual metric. Show that $f : Y \to Y$ defined by

$$f(x) = \frac{1}{x} \ ; \ x \in \mathbb{R} \setminus \{0\} \ ;$$
$$f(0) = y \qquad ; \qquad f(y) = 0$$

is a continuous function on Y.

4. Given two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on a set A, with $\mathcal{T}_1 \subset \mathcal{T}_2$, prove that if $\{A, \mathcal{T}_2\}$ is compact then so is $\{A, \mathcal{T}_1\}$.

5. Which of the following subsets of $\mathbb{R}$, $\mathbb{R}^2$ are compact?

(i)	$[0, 1)$
(ii)	$[0, \infty)$
(iii)	$\mathbb{Q} \cap [0, 1]$
(iv)	$\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$
(v)	$\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\}$
(vi)	$\{(x, y) \in \mathbb{R}^2 \ : \ \|x\| + \|y\| \leq 1\}$
(vii)	$\{(x, y) \in \mathbb{R}^2 \ : \ x \geq 1, 0 \leq y \leq 1/x\}$

6. Let A be a non-empty compact subset of the metric space (X, d), let L be any fixed positive real number, and let $F : A \to \mathbb{R}$ be a function with the property that

$$|F(x) - F(y)| \leq L d(x, y) \ \forall \ x, y \in A \ .$$

Show that for any sequence $a_1, a_2, a_3, \ldots$ in A, the real sequence $F(a_1), F(a_2), F(a_3), \ldots$ has a convergent subsequence with limit $F(a_0)$ for some $a_0 \in A$.

Deduce that the set $F(A) = \{F(a) : a \in A\}$ is compact in $\mathbb{R}$.

Hence show that there is $a^* \in A$ such that $F(a^*) \leq F(a)$ for $a \in A$.