1. In each of the following cases, say whether \((X, d)\) is a metric space or not. If it is not, say which of the axioms fails.

(a) \(X = \mathbb{R}^2\); \(d((x, y), (x', y')) = |y - y'|\).

(b) \(X = \mathbb{C}\); \(d(z_1, z_2) = |z_1 - z_2|\).

(c) \(X = \mathbb{Q}\); \(d(x, y) = (x - y)^3\).

(d) \(X = \mathbb{C}\); \(d(z_1, z_2) = \min\{|z_1| + |z_2|, |z_1 - 1| + |z_2 - 1|\}\) if \(z_1 \neq z_2\), \(d(z, z) = 0\).

(e) \(X = \mathbb{R}\); \(d(x, y) = \left|\int_x^y f(t)\, dt\right|\),

where \(f : \mathbb{R} \to \mathbb{R}\) is a given positive integrable function.

2. Sketch the sets \(\{x \in \mathbb{R}^2, d(x, 0) < 1\}\) when \(d(x, y)\) is
   (a) The Euclidean metric;
   (b) The taxicab metric;
   (c) The sup metric;
   (d) The discrete metric.

3. For \(z_j = x_j + iy_j \in \mathbb{C}\), let

\[
\xi_j = \frac{2x_j}{1 + |z_j|^2} ; \quad \eta_j = \frac{2y_j}{1 + |z_j|^2} ; \quad \zeta_j = \frac{1 - |z_j|^2}{1 + |z_j|^2}.
\]

Show that

(i) \(\xi_j^2 + \eta_j^2 + \zeta_j^2 = 1\)

(ii) \(\left((\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2\right)^{1/2} = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}\)

Hence deduce that

\[
d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}
\]

is a metric on \(\mathbb{C}\).

Show that the sequence \(\{z_n = n\}\) is a Cauchy sequence with respect to this metric.

4. Show that if for some \(x_0 \in S\), \(d(x, x_0) < k\) for all \(x \in Q\), then for any \(a \in S\) there is a constant \(k_a\) such that \(d(x, a) < k_a\) for all \(x \in Q\).

(That is, a set \(Q\) is bounded in \(S\) if it is bounded with respect to some member of \(S\).)