NEW TUTORIAL 3.1 FOR MATH3402

Question 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a Cauchy sequence in a metric space in* (X, d) *and let* $\{a_{i_k}\}_{n=1}^{\infty}$ *be a subsequence of* $\{a_n\}_{n=1}^{\infty}$*. Show that*

$$\lim_{n\to\infty} d(a_n, a_{i_n}) = 0.$$

Solution. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, for any $\varepsilon > 0$, there is an integer $n_0 \in \mathbb{N}$ such that for $n, m > n_0$ we have $d(a_n, a_m) < \varepsilon$.

Now $i_n \geq n > n_0$ and therefore

$$d(a_n, a_{i_n}) < \varepsilon$$

This means that $\lim_{n\to\infty} d(a_n, a_{i_n}) = 0$. $\square$

Question 2. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a Cauchy sequence in a metric space in* (X, d) *and let* $\{a_{i_n}\}_{n=1}^{\infty}$ *be a subsequence of* $\{a_n\}_{n=1}^{\infty}$ *converging to* $p \in X$*. Show that* $\{a_n\}_{n=1}^{\infty}$ *also converges to* p*.*

Solution. By the triangle inequality,

$$d(a_n, p) \leq d(a_n, a_{i_n}) + d(a_{i_n}, p)$$

and therefore

$$\lim_{n\to\infty} d(a_n, p) \leq \lim_{n\to\infty} d(a_n, a_{i_n}) + \lim_{n\to\infty} d(a_{i_n}, p).$$

By the question 1, we know $\lim_{n\to\infty} d(a_n, a_{i_n}) = 0$. Then

$$\lim_{n\to\infty} d(a_n, p) = 0 \quad \text{and so } a_n \to p. \quad \square$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Question 3. *Let $\{b_n\}_{n=1}^{\infty}$ be a Cauchy sequence in a metric space X, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence in X such that $d(a_n, b_n) < \frac{1}{n}$ for every $n \in \mathbb{N}$.*

(i) Show that $\{a_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in X.

(ii)Show that $\{a_n\}_{n=1}^{\infty}$ converges to $p \in X$ if and only if $\{b_n\}_{n=1}^{\infty}$ converges to $p \in X$.

Solution. (i) Since $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\forall \varepsilon > 0$, there is a $N \in \mathbb{N}$ such that for all integers $n, m \geq N$, we have

$$d(b_m, b_n) < \frac{1}{3\varepsilon}.$$

There is N_1 such that $\frac{1}{n} \leq \frac{1}{3\varepsilon}$ for $n \geq N_1$. Choose $N_2 = \max\{N, N_1\}$. Then for $n, m \geq N_2$

$$d(a_m, a_n) \leq d(a_m, b_m) + d(b_n, a_n) + d(b_m, b_n) < \varepsilon.$$

(ii) Suppose that $\lim_{n\to\infty} a_n = p$.

$$d(b_n, p) \leq d(b_n, a_n) + d(a_n, p) \leq \frac{1}{n} + d(a_n, p)$$

As $n \to \infty$, $\lim_{n\to\infty} d(b_n, p) = 0$,so $\lim_{n\to\infty} b_n = p$. □

Question 4. *Let A be a subset of a metric space (X, d). Then $x \in X$ is a point of accumulation (limit point) of A if and only if there is an infinite sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in A$ which converges to x.*

Proof. Let $x \in X$ be the limit of the infinite sequence $\{a_n\}$ and G an open set od X which contains x. Then there is an $\varepsilon > 0$ with $B(x; \varepsilon) \subset G$. But there is an $N \in \mathbb{N}$ with $a_n \in B_\varepsilon(x)$ whenever $n > N$. At least one of these a_n must be a distict from x and so $A \cap B_\varepsilon(x) \backslash \{x\} \neq \emptyset$, this shows that $A \cap G \backslash \{x\} \neq \emptyset$. Thus x is a point of accumulation of A.

Conversely, let x be a point of accumulation of A. Then choose $a_0 \in A \cap B_1(x)$, with $a_0 \neq x$. Suppose that for $n \in \mathbb{N}$, a_0, ..., a_n have chosen from A which are different. Put $r_n = \min\{d(a_1, x), ..., d(x_n, x), \frac{1}{n+1}\}$. Then $B_{r_n}(x)$ is an open set X contains x and hence there exists a point $a_{n+1} \neq x$ of A. Moreover, $a_{n+1} \neq a_j$ for $0 \leq j \leq n$ by the definition of r_n. The sequence $\{a_n\}$ converges to x. □