Question 1. Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence in a metric space in \( (X,d) \) and let \( \{a_{i_n}\}_{n=1}^{\infty} \) be a subsequence of \( \{a_n\}_{n=1}^{\infty} \). Show that
\[
\lim_{n \to \infty} d(a_n, a_{i_n}) = 0.
\]

Solution. Since \( \{a_n\}_{n=1}^{\infty} \) is a Cauchy sequence, for any \( \varepsilon > 0 \), there is an integer \( n_0 \in \mathbb{N} \) such that for \( n, m > n_0 \) we have \( d(a_n, a_m) < \varepsilon \).

Now \( i_n \geq n > n_0 \) and therefore
\[
d(a_n, a_{i_n}) < \varepsilon
\]
This means that \( \lim_{n \to \infty} d(a_n, a_{i_n}) = 0 \). \( \Box \)

Question 2. Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence in a metric space in \( (X,d) \) and let \( \{a_{i_n}\}_{n=1}^{\infty} \) be a subsequence of \( \{a_n\}_{n=1}^{\infty} \) converging to \( p \in X \). Show that \( \{a_n\}_{n=1}^{\infty} \) also converges to \( p \).

Solution. By the triangle inequality,
\[
d(a_n, p) \leq d(a_n, a_{i_n}) + d(a_{i_n}, p)
\]
and therefore
\[
\lim_{n \to \infty} d(a_n, p) \leq \lim_{n \to \infty} d(a_n, a_{i_n}) + \lim_{n \to \infty} d(a_{i_n}, p).
\]
By the question 1, we know \( \lim_{n \to \infty} d(a_n, a_{i_n}) = 0 \). Then
\[
\lim_{n \to \infty} d(a_n, p) = 0 \quad \text{and so} \quad a_n \to p. \quad \Box
\]
Question 3. Let \( \{b_n\}_{n=1}^{\infty} \) be a Cauchy sequence in a metric space \( X \), and let \( \{a_n\}_{n=1}^{\infty} \) be a sequence in \( X \) such that \( d(a_n, b_n) < \frac{1}{n} \) for every \( n \in \mathbb{N} \).

(i) Show that \( \{a_n\}_{n=1}^{\infty} \) is also a Cauchy sequence in \( X \).

(ii) Show that \( \{a_n\}_{n=1}^{\infty} \) converges to \( p \in X \) if and only if \( \{b_n\}_{n=1}^{\infty} \) converges to \( p \in X \).

Solution. (i) Since \( \{b_n\}_{n=1}^{\infty} \) is a Cauchy sequence, \( \forall \varepsilon > 0 \), there is a \( N \in \mathbb{N} \) such that for all integers \( n, m \geq N \), we have

\[
d(b_m, b_n) < \frac{1}{3\varepsilon}.
\]

There is \( N_1 \) such that \( \frac{1}{n} \leq \frac{1}{3\varepsilon} \) for \( n \geq N_1 \). Choose \( N_2 = \max\{N, N_1\} \). Then for \( n, m \geq N_2 \)

\[
d(a_m, a_n) \leq d(a_m, b_m) + d(b_n, a_n) + d(b_m, b_n) < \varepsilon.
\]

(ii) Suppose that \( \lim_{n \to \infty} a_n = p \).

\[
d(b_n, p) \leq d(b_n, a_n) + d(a_n, p) \leq \frac{1}{n} + d(a_n, p)
\]

As \( n \to \infty \), \( \lim_{n \to \infty} d(b_n, p) = 0 \), so \( \lim_{n \to \infty} b_n = p \). □

Question 4. Let \( A \) be a subset of a metric space \( (X, d) \). Then \( x \in X \) is a point of accumulation (limit point) of \( A \) if and only if there is an infinite sequence \( \{a_n\}_{n=1}^{\infty} \) with \( a_n \in A \) which converges to \( x \).

Proof. Let \( x \in X \) be the limit of the infinite sequence \( \{a_n\} \) and \( G \) an open set of \( X \) which contains \( x \). Then there is an \( \varepsilon > 0 \) with \( B(x; \varepsilon) \subset G \). But there is an \( N \in \mathbb{N} \) with \( a_n \in B_\varepsilon(x) \) whenever \( n > N \). At least one of these \( a_n \) must be a distinct from \( x \) and so \( A \cap B_\varepsilon(x) \setminus \{x\} \neq \emptyset \), this shows that \( A \cap G \setminus \{x\} \neq \emptyset \). Thus \( x \) is a point of accumulation of \( A \).

Conversely, let \( x \) be a point of accumulation of \( A \). Then choose \( a_0 \in A \cap B_1(x) \), with \( a_0 \neq x \). Suppose that for \( n \in \mathbb{N} \), \( a_0, ..., a_n \) have chosen from \( A \) which are different. Put \( r_n = \min\{d(a_1, x), ..., d(x_n, x), \frac{1}{n+1}\} \). Then \( B_{r_n}(x) \) is an open set \( X \) contains \( x \) and hence there exists a point \( a_{n+1} \neq x \) of \( A \). Moreover, \( a_{n+1} \neq a_j \) for \( 0 \leq j \leq n \) by the definition of \( r_n \). The sequence \( \{a_n\} \) converges to \( x \). □