MATH 3402
TUTORIAL SHEET 8
SOLUTIONS

1. Which of the following are contractions on $\mathbb{R}^2$ with the Euclidean metric?

(i) $f: R^2 \rightarrow R^2$ given by the matrix $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

(ii) $f: R^2 \rightarrow R^2$ given by the matrix $\begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

(iii) $f: R^2 \rightarrow R^2$ given by the matrix $\begin{pmatrix} \frac{1}{2} & \frac{6}{7} \\ -\frac{5}{6} & \frac{2}{3} \end{pmatrix}$

Ans. Ax will be a contraction if $||Ax|| \leq k < 1$ when $||x|| = 1$.
Therefore take $x = (\cos t, \sin t)'$.

(i)
$$Ax = \begin{pmatrix} \frac{1}{2}\cos t \\ \frac{1}{3}\sin t \end{pmatrix}$$
$$||Ax||^2 = \frac{1}{4}\cos^2 t + \frac{1}{9}\sin^2 t \leq \frac{1}{4}$$
$$||Ax|| \leq \frac{1}{2}$$

therefore this is a contraction.

(ii)
$$Ax = \begin{pmatrix} \frac{1}{6}\cos t + \frac{1}{6}\sin t \\ -\frac{1}{3}\cos t + \frac{2}{3}\sin t \end{pmatrix}$$
$$||Ax||^2 = \frac{1}{36}\cos^2 t + \frac{1}{18}\sin t\cos t + \frac{1}{36}\sin^2 t + \frac{1}{9}\cos^2 t - \frac{1}{9}\sin t\cos t + \frac{4}{9}\sin^2 t$$
$$= \frac{5}{36}\cos^2 t - \frac{1}{18}\sin t\cos t + \frac{17}{36}\sin^2 t = \frac{11}{36} - \frac{1}{6}\cos 2t - \frac{1}{36}\sin 2t$$
$$\leq \frac{11}{36} + \frac{1}{6} + \frac{1}{36} = \frac{1}{2}$$
$$||Ax|| \leq \frac{1}{\sqrt{2}}$$

therefore this is a contraction.

(iii)
$$Ax = \begin{pmatrix} \frac{1}{2}\cos t + \frac{6}{7}\sin t \\ -\frac{5}{6}\cos t + \frac{2}{3}\sin t \end{pmatrix}$$
$$||Ax||^2 = \frac{1}{4}\cos^2 t + \frac{6}{7}\cos t\sin t + \frac{36}{49}\sin^2 t + \frac{25}{36}\cos^2 t - \frac{10}{9}\cos t\sin t + \frac{4}{9}\sin^2 t$$
$$= \frac{34}{36}\cos^2 t - \frac{16}{63}\cos t\sin t + \frac{520}{441}\sin^2 t$$
$$= \frac{520}{441} > 1 \text{ when } \cos t = 0$$

therefore this is not a contraction.

2. Show that if $h \in C(a,b)$ (with the uniform metric) and $b-a<1$, then $\mathcal{F} : C(a,b) \to C(a,b)$ is a contraction mapping, where

$$\mathcal{F}(g)(x) = h(x) + \int_a^x g(t)\,dt$$

for $x \in [a,b]$, $g \in C(a,b)$.

Ans.

$$\begin{aligned} |\mathcal{F}(\phi) - \mathcal{F}(\psi)| = & \left| \int_a^x (\phi(t) - \psi(t))\,dt \right| \\ \le & \int_a^x |\phi(t) - \psi(t)|\,dt \\ \le & (x-a)||\phi - \psi|| \le (b-a)||\phi - \psi|| \\ ||\mathcal{F}(\phi) - \mathcal{F}(\psi)|| \le & (b-a)||\phi - \psi|| \end{aligned}$$

therefore $\mathcal{F}$ is a contraction mapping on $C(a,b)$.

What is the fixed point of $\mathcal{F}$.

Ans While **you cannot assume that** $h(x)$ **is differentiable,** if

$$g(x) = h(x) + \int_a^x g(t)\,dt$$

then

$$\phi(x) = (g(x) - h(x)) = \int_a^x g(t)\,dt$$

is differentiable, and

$$\begin{aligned} \phi'(x) = g(x) = \phi(x) + h(x)\ ; \phi(a) = 0 \\ \phi(x) = \int_a^x e^{x-t} h(t)\,dt \\ g(x) = h(x) + \int_a^x e^{x-t} h(t)\,dt \end{aligned}$$

3. The space ℓ^1 is defined to be the set of all sequences $x = \{\xi^{(i)}\}$ such that $\sum_{i=1}^{\infty} |\xi^{(i)}|$ converges.

If $b = \{\beta^{(i)}\}$ and $c = \{\gamma^{(i)}\}$ are elements of ℓ^1, show that if $|\lambda|$ is sufficiently small, there is a unique element $a = \{\alpha^{(i)}\} \in \ell^1$ such that

$$\alpha^{(n)} = \beta^{(n)} + \lambda \sum_{i=1}^{\infty} \gamma^{(i)} \alpha^{(n+i-1)}\ .$$

Ans. Consider the mapping from ℓ^1 to ℓ^1 defined by

$$\mathcal{T}(x) = b + \lambda C x$$

where C is the infinite matrix

$$C = \begin{pmatrix} \gamma^{(1)} & \gamma^{(2)} & \gamma^{(3)} & \dots \\ 0 & \gamma^{(1)} & \gamma^{(2)} & \dots \\ 0 & 0 & \gamma^{(1)} & \dots \\ . & . & . & \dots \\ . & . & . & \dots \end{pmatrix} .$$

Then

$$\begin{aligned} ||\mathcal{T}(x) - \mathcal{T}(y)|| &= |\lambda| \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \gamma^{(j)} (\xi^{(i+j-1)} - \eta^{(i+j-1)} \right| \\ &\le |\lambda| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\gamma^{(j)}| \, |\xi^{(i+j-1)} - \eta^{(i+j-1)}| \\ (*) \qquad &= |\lambda| \sum_{j=1}^{\infty} |\gamma^{(j)}| \sum_{i=1}^{\infty} |\xi^{(i+j-1)} - \eta^{(i+j-1)}| \\ &\le |\lambda| \, ||c||_1 \, ||x - y||_1 \end{aligned}$$

Therefore, if $|\lambda| < 1/||c||$, $\mathcal{T}$ is a contraction mapping on ℓ^1 which is a complete space, and the problem has a unique solution in ℓ^1.

(*) This step is justified since all the terms in the double sum are positive, and therefore the convergence is absolute.

4. Let T be the linear transformation from X to Y whose matrix representation is

$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} .$$

Determine $||T||$ when

(a) $X = \ell^2(2)$ and $Y = \ell^2(2)$;

Ans.

$$A^*A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$|tI - A| = \begin{vmatrix} t-1 & -2 \\ -2 & t-5 \end{vmatrix} = t^2 - 6t + 1$$

$$\lambda_1 = 3 + 2\sqrt{2} \, ; ||A||_2 = 1 + \sqrt{2}$$

(b) $X = \ell^1(2)$ and $Y = \ell^1(2)$;

Ans. $||A||_1 = \max(1, 3) = 3$.

(c) $X = \ell^\infty(2)$ and $Y = \ell^\infty(2)$;

Ans. $||A||_\infty = \max(3, 1) = 3$.

(d) $X = \ell^1(2)$ and $Y = \ell^\infty(2)$;

Ans. Consider $x = (1-t, t)'$, $0 \le t \le 1$. $||x||_1 = 1$,

$$Ax = \begin{pmatrix} 1+t \\ t \end{pmatrix} \quad ||Ax||_\infty = 1 + t \le 2$$

Therefore $||A|| = 2$.