MATH 3402

TUTORIAL SHEET 7

SOLUTIONS

1. The function $f : (A, d) \to (A, d)$ has the property that

$$d(f(x), f(y)) < d(x, y) \ \forall \ x \neq y \in A \ .$$

Show that if f has a fixed point in A then it is unique.

Ans. Suppose that a and b are fixed points of f in A.
Then

$$d(a, b) = d(f(a), f(b)) < d(a, b) \text{ if } a \neq b \ .$$

Since this is impossible, we must have $a = b$.
(Note that this does not prove that a fixed point actually exists.)

2. The real function f is continuous from $[a, b]$ to $[a, b]$.
Show that f has a fixed point in $[a, b]$.

Ans. If either $f(a) = a$ or $f(b) = b$ there is nothing to prove.
Therefore assume $f(a) > a$ and $f(b) < b$.
The function $\phi(x) = x - f(x)$ is continuous on $[a, b]$, and

$$\phi(a) < 0 \ ; \ \phi(b) > 0 \ .$$

Therefore, by the Intermediate Value Theorem there is $c \in (a, b)$ such that

$$\phi(c) = 0 \ ; \ f(c) = c \ .$$

3. Determine to 3dp the solution of the equation

$$x^3 = 6x + 6 \ .$$

(The solution is approximately 3.)

Ans. If $F(x) = x^3 - 6x - 6$, $F'(x) = 3x^2 - 6$, $F'(3) = 21$.
Therefore consider

$$f(x) = x - \frac{1}{21} F(x) = \frac{1}{21}(6 + 27x - x^3)$$

with $x_0 = 3$.

$$\begin{aligned} x_1 &= 2.857 \\ x_2 &= 2.849 \\ x_3 &= 2.847 \\ x_4 &= 2.847 \end{aligned}$$

4. For fixed $a \in \mathbb{R}$, we define the mapping from $\mathbb{R} \setminus \{-1\}$ to $\mathbb{R}$ by

$$f(x) = 1 + \frac{a}{1+x} \ .$$

(a) For which values of a does this mapping have a fixed point?

Ans: If $f(x) = x$, then

$$\begin{aligned} x &= 1 + \frac{a}{1+x} \\ (x-1)(x+1) &= a \\ x^2 &= a+1 \\ x &= \pm\sqrt{a+1} \end{aligned}$$

which requires $a \geq -1$.

(b) For which values of a is the mapping a (local) contraction mapping?

Ans:

$$f'(x) = -\frac{a}{(1+x)^2}$$

When $x = \sqrt{a+1}$,

$$\begin{aligned} |f'(\sqrt{a+1})| &= \frac{|a|}{(1+\sqrt{a+1})^2} \\ &= \frac{|\sqrt{a+1}-1|}{\sqrt{a+1}+1} \\ &< 1 \text{ if } a > -1 \end{aligned}$$

When $x = -\sqrt{a+1}$,

$$\begin{aligned} |f'(-\sqrt{a+1})| &= \frac{|a|}{(1-\sqrt{a+1})^2} \\ &= \frac{\sqrt{a+1}+1}{|\sqrt{a+1}-1|} \\ &\geq 1 \end{aligned}$$

Therefore we get a contraction mapping if $a > -1$. The sequence $\{x_{n+1} = f(x_n)\}$ converges to $\sqrt{a+1}$.

(c) Starting with $w_0 = 1$, for which values of a does the sequence generated by

$$w_{n+1} = f(w_n)$$

converge?

Ans: In addition to the values mentioned above, the sequence also converges for $a = -1$, albeit slowly.

The sequence generates the continued fraction expansion

$$\sqrt{1+a} = 1 + \frac{a}{2+}\ \frac{a}{2+}\ \frac{a}{2+}\ \ldots$$

(d)If we choose $a \in \mathbb{C}$ instead, show that the function is a contraction mapping provided $Re(\sqrt{1+a}) \geq \epsilon > 0$.

Ans: Let $\sqrt{1+a} = \alpha + i\beta$, where $\alpha \geq \epsilon$.

$$\begin{aligned}\left|f'(\sqrt{1+a})\right|^2 &= \frac{|\alpha + i\beta - 1|^2}{|\alpha + i\beta + 1|^2} \\ &= \frac{(\alpha-1)^2 + \beta^2}{(\alpha+1)^2 + \beta^2} \\ &= 1 - \frac{4\alpha}{(\alpha^2+1)^2 + \beta^2} < 1\end{aligned}$$

5. Let $C(0, \frac{1}{2})$ be the set of real functions continuous on $[0, \frac{1}{2}]$, together with the uniform sup metric.

Define $f : C(0, \frac{1}{2}) \to C(0, \frac{1}{2})$ by

$$(f(x))(t) = t(x(t) + 1) \ .$$

Show that f is a contraction mapping, and determine its fixed point.

Ans.

$$\begin{aligned}||f(x) - f(y)|| &= \sup_{0 \leq t \leq \frac{1}{2}} |tx - ty| \\ &\leq \frac{1}{2} \sup_{0 \leq t \leq \frac{1}{2}} |x - y| = \frac{1}{2}||x - y||\end{aligned}$$

If $x = tx + t$, $(1-t)x = t$,

$$x = \frac{t}{1-t} \ .$$

Verify that the sequence starting with $x_0(t) = t$ converges to the fixed point.
Ans.

$$x_0 = t$$
$$x_1 = t^2 + t$$
$$x_2 = t^3 + t^2 + t$$
$$x_n = \sum_{i=1}^{n+1} t^i = \frac{t - t^{n+2}}{1-t}$$
$$\left\| x_n - \frac{t}{1-t} \right\| = \left\| \frac{t^{n+2}}{1-t} \right\| \leq 2\left(\frac{1}{2}\right)^{n+2} \to 0 \text{ as } n \to \infty$$