MATH 3402

TUTORIAL SHEET 6

SOLUTIONS

1. Show that $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

Ans.

If $x \in Cl(A)$, then every neighbourhood of x contains a point of A, and therefore of $A \cup B$. Therefore $Cl(A) \subset Cl(A \cup B)$

Similarly, $Cl(B) \subset Cl(A \cup B)$, and hence $Cl(A) \cup Cl(B) \subset Cl(A \cup B)$.

If $x \in Cl(A \cup B)$, then every neighbourhood of x contains a point of $A \cup B$ and therefore it contains a point of $Cl(A) \cup Cl(B)$.

Therefore $x \in Cl(Cl(A) \cup Cl(B))$, and $Cl(A \cup B) \subset Cl(Cl(A) \cup Cl(B))$.

But $Cl(A) \cup Cl(B)$ is closed, so that $Cl(Cl(A) \cup Cl(B)) = Cl(A) \cup Cl(B)$.

Hence $Cl(A \cup B) \subset Cl(A) \cup Cl(B)$, and so the sets are equal.

2. Verify that the collection $\mathcal{T}_Y$ defined for the one-point compactification process is a topology on Y.

Ans. We have already seen that $Y \in \mathcal{T}_Y$, and since $\phi \in \mathcal{T}_X$, $\phi \in \mathcal{T}_Y$ also.

If U_1 and U_2 are in $\mathcal{T}_Y$, then

(a) if they are both in $\mathcal{T}_X$,

$$U_1 \cap U_2 \in \mathcal{T}_X \subset \mathcal{T}_Y$$

(b) if U_1 is in $\mathcal{T}_X$ and $U_2 = W_2 \cup \{y\}$, then

$$U_1 \cap U_2 = U_1 \cap W_2 \in \mathcal{T}_X \subset \mathcal{T}_Y$$

(c) if neither U_1 nor U_2 is in $\mathcal{T}_X$, then $U_1 = W_1 \cup \{y\}$ and $U_2 = W_2 \cup \{y\}$ where $X \setminus W_1$ and $X \setminus W_2$ are compact in X. Therefore

$$U_1 \cap U_2 = (W_1 \cap W_2) \cup \{y\} = W_3 \cup \{y\}$$

where $X \setminus W_3 = (X \setminus W_1) \cup (X \setminus W_2)$ is compact in X. Therefore $U_1 \cap U_2 \in \mathcal{T}_Y$.

For

$$\cup_i U_i \ ,$$

if all the sets are in $\mathcal{T}_X$ then so is their union, and hence the union is in $\mathcal{T}_X \subset \mathcal{T}_Y$.

Otherwise

$$\cup_i U_i = (\cup_i (U_i \cap X)) \cup \{y\} = W \cup \{y\}$$

where $W \in \mathcal{T}_X$ and at least one of the sets $U_i \cap X$ has a compact complement in X. Denote this set by W'.

$X \setminus W$ is closed in X, and is a subset of $X \setminus W'$. Therefore $X \setminus W$ is compact in X, and $W \cup \{y\} \in \mathcal{T}_Y$.

3. Let $Y = \mathbb{R} \cup \{y\}$ be the one-point compactification of $\mathbb{R}$ with the usual metric. Show that $f : Y \to Y$ defined by

$$f(x) = \frac{1}{x} \; ; \; x \in \mathbb{R} \setminus \{0\} \; ;$$
$$f(0) = y \qquad ; \qquad f(y) = 0$$

is a continuous function on Y.

Ans. The set of all open intervals in $\mathbb{R}$ form a basis for $\mathbb{R}$.

We can extend this to a basis for Y by adding all open sets of the form $(-\infty, b) \cup \{y\} \cup (a, \infty)$ where $b \leq 0 \leq a$.

Then

$$f^{-1}((a,b)) = (\frac{1}{b}, \frac{1}{a}) \text{ if } 0 < a < b \text{ or } a < b < 0$$

$$f^{-1}((a,b)) = (-\infty, \frac{1}{a}) \cup \{y\} \cup (\frac{1}{b}, \infty) \text{ if } a < 0 < b$$

and

$$f^{-1}((-\infty, b) \cup \{y\} \cup (a, \infty)) = (\frac{1}{b}, \frac{1}{a})$$

where $\frac{1}{b} = -\infty$ if $b = 0$ and $\frac{1}{a} = \infty$ if $a = 0$.

Since the inverse images of the basis sets are open in Y, f is continuous.

4. Given two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on a set A, with $\mathcal{T}_1 \subset \mathcal{T}_2$, prove that if $(A, \mathcal{T}_2)$ is compact then so is $(A, \mathcal{T}_1)$.

Ans. Since $\mathcal{T}_1 \subset \mathcal{T}_2$, any open cover of A by elements of $\mathcal{T}_1$ is an open cover by elements of $\mathcal{T}_2$.

Therefore there is a finite subcover, and $(A, \mathcal{T}_1)$ is compact.

5. Which of the following subsets of $\mathbb{R}$, $\mathbb{R}^2$ are compact?

(i)	$[0,1)$
(ii)	$[0,\infty)$
(iii)	$\mathbb{Q} \cap [0,1]$
(iv)	$\{(x,y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = 1\}$
(v)	$\{(x,y) \in \mathbb{R}^2 \, : \, x^2 + y^2 < 1\}$
(vi)	$\{(x,y) \in \mathbb{R}^2 \, : \, \lvert x\rvert + \lvert y\rvert \leq 1\}$
(vii)	$\{(x,y) \in \mathbb{R}^2 \, : \, x \geq 1, 0 \leq y \leq 1/x\}$

Ans.
(i) Not closed therfore not compact.
(ii) Not bounded therefore not compact.
(iii) Not closed therefore not compact.
(iv) Closed and bounded, therefore compact.
(v) Not closed therefore not compact.
(vi) Closed and bounded, therefore compact.
(vii) Not bounded therefore not compact.

6. Let A be a non-empty compact subset of the metric space (X, d), let L be any fixed positive real number, and let $F : A \to \mathbb{R}$ be a function with the property that

$$|F(x) - F(y)| \leq Ld(x, y) \ \forall \ x, y \in A \ .$$

Show that for any sequence $a_1, a_2, a_3, \ldots$ in A, the real sequence

$$F(a_1), F(a_2), F(a_3), \ldots$$

has a convergent subsequence with limit $F(a_0)$ for some $a_0 \in A$.

Deduce that the set $F(A) = \{F(a) : a \in A\}$ is compact in $\mathbb{R}$.

Hence show that there is $a^* \in A$ such that $F(a^*) \leq F(a)$ for $a \in A$.

Ans. Since A is a compact set in a metric space, it is bounded.

That is, for fixed $a \in A$, $d(x, a) \leq K$ for all $x \in A$.

Therefore $|F(x) - F(a)| \leq LK$ for all $F(x) \in F(A)$, and $F(A)$ is bounded in $\mathbb{R}$.

It follows from the Bolzano Weierstrass theorem that the real sequence $\{F(a_i)\}$ has a limit point in $\mathbb{R}$.

We can choose a subsequence $\{a_{i_n}\}$ such that $\{F(a_{i_n})\}$ converges to this limit.

This subsequence is an infinite subset in a compact set, so that it has a limit point $a_0 \in A$.

Given any $\epsilon > 0$, there are infinitely many term of the subsequence such that $d(a_{i_n}, a_0) < \frac{\epsilon}{L}$.

For these terms we have $|F(a_{i_n}) - F(a_0)| < \epsilon$ so that the limit of the convergent subsequence is $F(a_0)$.

The same argument shows that any limit point of $F(A)$ is in $F(A)$, so that $F(A)$ is closed.

By the Heine-Borel Theorem, $F(A)$ is compact.

(Alternatively, we could note that F is uniformly continuous on A.)

Since $F(A)$ is bounded it has a greatest lower bound, and since $F(A)$ is closed this bound is in the set.

That is, there exists $a^* \in A$ such that $F(a^*) \leq F(a); a \in A$.