MATH 3402
Tutorial sheet 5
Solutions

1. If f is a many-one transformation of A into B, and A_1 and A_2 are subsets of A, prove that

(a) $$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)\ ;$$
(b) $$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)\ .$$

In the second case, show that equality holds for all A_1 and A_2 if and only if f is a one-one transformation.

Ans: If $x \in A_1 \cup A_2$, then $x \in A_1$ or $x \in A_2$.
Therefore, $f(x) \in f(A_1)$ or $f(x) \in f(A_2)$.
This means $f(A_1 \cup A_2) \subset f(A_1) \cup f(A_2)$.

If $y \in f(A_1) \cup f(A_2)$, then $y \in f(A_1)$ or $y \in f(A_2)$.
Therefore $y = f(x)$ where $x \in A_1$ or $x \in A_2$; i.e. $x \in A_1 \cup A_2$.
This means that $f(A_1) \cup f(A_2) \subset f(A_1 \cup A_2)$.

Together these results mean $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

If $x \in A_1 \cap A_2$, then $x \in A_1$ and $x \in A_2$.
Therefore, $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$.
This means that $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

If f is many-one, then there are at least two distinct elements $x_1 \neq x_2$ such that $f(x_1) = f(x_2)(= y$ say).
Setting $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$, we see that

$$A_1 \cap A_2 = \phi\ ; f(A_1 \cap A_2) = \phi$$

while

$$f(A_1) \cap f(A_2) = \{y\} \neq \phi\ .$$

However, if f is 1-1, then $f^{-1}(y)$ is uniqely defined, and if $y \in f(A_1) \cap f(A_2)$, y is in both $f(A_1)$ and $f(A_2)$.
Therefore $f^{-1}(y)$ is in both A_1 and A_2, so that

$$f(A_1) \cap f(A_2) \subset f(A_1 \cap A_2)\ .$$

Combined with the first result this gives equality.

2. Let A be the set of real numbers, and let a subset of A be called open if it is A or the null set or if it consists of points x such that $x > k$ for some k.
Prove that the open sets defined in this way form a topology for A.

Ans: Denote the interval (k, ∞) by S_k. In this notation, $A = S_{-\infty}$ and $\phi = S_\infty$. Denote by $\mathcal{T}$ this collection of sets.

We need to show that $\mathcal{T}$ satisfies the axioms of a topology.

By construction, $\phi \in \mathcal{T}$ and $A \in \mathcal{T}$.

Consider the intersection of a finite number of sets in $\mathcal{T}$.

If ϕ is one of these, then the intersection is also ϕ which is an element of $\mathcal{T}$.

Otherwise, let the sets be $\{S_{\alpha_1}, \ldots, S_{\alpha_n}\}$.

If α_I is the maximum of $\{\alpha_1, \ldots, \alpha_n\}$, then

$$I = \bigcap_{i=1}^{n} S_{\alpha_i} = S_{\alpha_I}$$

since, if $x \in I$, $x \in S_{\alpha_I}$, while if $x \in S_{\alpha_I}$, then $x > \alpha_I$, so that $x > \alpha_i$ for $i = 1 \ldots n$. Therefore $x \in S_{\alpha_i}$ for all i and hence $x \in I$.

Consider the union of an arbitrary number of sets in $\mathcal{T}$.

Let the sets be S_α.

If the set of values $\{\alpha\}$ is not bounded below, then for any $x \in A$, there is $\alpha^* < x$, and $x \in S_{\alpha^*}$. Therefore the union of the sets is A which is in $\mathcal{T}$.

Otherwise, let $a = \inf\{\alpha\}$. If $x > a$, then x is not a lower bound for $\{\alpha\}$, so that for some α^*, $x > \alpha^* \geq a$.

Therefore

$$S_a \subset \bigcup_\alpha S_\alpha$$

On the other hand $\alpha \geq a$ for all α, so that

$$S_\alpha \subset S_a \ ; \bigcup_\alpha S_\alpha \subset S_a$$

Combining these results, we see that the union is $S_a \in \mathcal{T}$.

Since $\mathcal{T}$ satisfies the axioms, it is a topology for A.

3. If $M_1 = (A_1, d_1)$ and $M_2 = (A_2, d_2)$ are two metric spaces, show that the function defined by

$$d\left((x_1, x_2),\, (y_1, y_2)\right) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

where $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$ is a metric on $A_1 \times A_2$.

Show that the topology generated by this metric is the product topology.

Ans: d satisfies the axioms fora metric;

(a)
$$\begin{aligned} d((y_1, y_2), (x_1, x - 2)) &= d_1(y_1, x_1) + d_2(y_2, x_2) \\ &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &= d((x_1, x_2), (y_1, y_2)) \end{aligned}$$

(b)
$$\begin{gathered} d_1(x_1, y_1) \geq 0 \ , \ d_2(x_2, y_2) \geq 0 \ , \\ \text{therefore } d((x_1, x_2), (y_1, y_2)) \geq 0 \ . \end{gathered}$$

(c)
$$\begin{gathered} d((x_1, x_2), (y_1, y_2)) = 0 \text{ iff } d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0 \\ \text{i.e. iff } x_1 = y_1 \text{ and} x_2 = y_2 \end{gathered}$$

(d)
$$\begin{aligned} d((x_1,x_2),(z_1,z_2)) &= d_1(x_1,z_1)+d_2(x_2,z_2) \\ &\le d_1(x_1,y_1)+d_1(y_1,z_1)+d_2(x_2,y_2)+d_2(y_2,z_2) \\ &= d((x_1,x_2),(y_1,y_2))+d((y_1,y_2),(z_1,z_2)) \end{aligned}$$

Let U be open in (A_1,d_1) and V be open in (A_2,d_2). If $(x_0,y_0)\in U\times V$, then $x_0\in U$ and $y_0\in V$. Therefore there are positive constants ϵ_1 and ϵ_2 such that

$$N_1=\{x\in A_1; d_1(x,x_0)<\epsilon_1\}\subset U$$

and

$$N_2=\{y\in A_2; d_2(y,y_0)<\epsilon_2\}\subset V$$

Then if $\epsilon=\min(\epsilon_1,\epsilon_2)$,

$$\{(x,y)\in A_1\times A_2; d((x,y),(x_0,y_0))<\epsilon\}\subset N_1\times N_2\subset U\times V$$

Therefore $U\times V$ is open in $A_1\times A_2$.

Since the product topology consists of unions of sets of this form, every set in the product topology is in the topology generated by d.

Conversely, if W is in the topology generated by d, then for each $(x_0,y_0)\in W$ there is $\epsilon>0$ such that

$$N=\{(x,y)\in A_1\times A_2; d((x,y),(x_0,y_0))<\epsilon\}\subset W$$

Let

$$N_1=\{x\in A_1; d_1(x,x_0)<\frac{1}{2}\epsilon\}\subset U$$

and

$$N_2=\{y\in A_2; d_2(y,y_0)<\frac{1}{2}\epsilon\}\subset V$$

Then $N_1\times N_2$ is in the product topology, and

$$N_1\times N_2\subset N\subset W$$

Since

$$W=\bigcup_{(x_0,y_0)\in W} N_1\times N_2$$

W is in the product topology. Therefore these two topologies are identical.

4. If S, T are topological spaces homeomorphic respectively to S', T', prove that $S \times T$ is homeomorphic to $S' \times T'$.

Ans: Since S is homeomorphic to S' there is a bi-continuous 1-1 map f from S to S'.

Similarly there is a bi-continuous 1-1 map g from T to T'.

Let F be the map from $S \times T$ to $S' \times T'$ defined by $F(x, y) = (f(x), g(y))$.

If $F(x_1, y_1) = F(x_2, y_2)$, then $f(x_1) = f(x_2)$ and $g(y_1) = g(y_2)$. Since f and g are $1 - 1$, $(x_1, y_1) = (x_2, y_2)$ and F is also $1 - 1$.

If W' is open in $S' \times T'$, then W' is the union of sets of the form $U' \times V'$, where U' is open in S' and V' is open in T'.

Let $U = f^{-1}(U')$ and $V = g^{-1}(V')$. These are open sets in S and T respectively.

Then $F^{-1}(U' \times V') = U \times V$ is open in $S \times T$.

Therefore

$$\begin{aligned} F^{-1}(W') &= F^{-1}\left(\bigcup U' \times V'\right) \\ &= \bigcup \left(F^{-1}(U' \times V')\right) \\ &= \bigcup U \times V \end{aligned}$$

is open in $S \times T$.

Therefore F is continuous from $S \times T$ to $S' \times T'$.

Similarly F^{-1} is continuous, so that F is a homeomorphism from $S \times T$ to $S' \times T'$.