1. If \( f \) is a many-one transformation of \( A \) into \( B \), and \( A_1 \) and \( A_2 \) are subsets of \( A \), prove that

(a) \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \);  
(b) \( f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2) \).

In the second case, show that equality holds for all \( A_1 \) and \( A_2 \) if and only if \( f \) is a one-one transformation.

**Ans:** If \( x \in A_1 \cup A_2 \), then \( x \in A_1 \) or \( x \in A_2 \).
Therefore, \( f(x) \in f(A_1) \) or \( f(x) \in f(A_2) \).
This means \( f(A_1 \cup A_2) \subset f(A_1) \cup f(A_2) \).

If \( y \in f(A_1) \cup f(A_2) \), then \( y \in f(A_1) \) or \( y \in f(A_2) \).
Therefore \( y = f(x) \) where \( x \in A_1 \) or \( x \in A_2 \); i.e. \( x \in A_1 \cup A_2 \).
This means that \( f(A_1) \cup f(A_2) \subset f(A_1 \cup A_2) \).
Together these results mean \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \).

If \( x \in A_1 \cap A_2 \), then \( x \in A_1 \) and \( x \in A_2 \).
Therefore, \( f(x) \in f(A_1) \) and \( f(x) \in f(A_2) \).
This means that \( f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2) \).

If \( f \) is many-one, then there are at least two distinct elements \( x_1 \neq x_2 \) such that \( f(x_1) = f(x_2) \)(= \( y \) say).
Setting \( A_1 = \{x_1\} \) and \( A_2 = \{x_2\} \), we see that

\[ A_1 \cap A_2 = \phi ; f(A_1 \cap A_2) = \phi \]

while

\[ f(A_1) \cap f(A_2) = \{y\} \neq \phi . \]

However, if \( f \) is 1-1, then \( f^{-1}(y) \) is uniquely defined, and if \( y \in f(A_1) \cap f(A_2) \), \( y \)
is in both \( f(A_1) \) and \( f(A_2) \).
Therefore \( f^{-1}(y) \) is in both \( A_1 \) and \( A_2 \), so that

\[ f(A_1) \cap f(A_2) \subset f(A_1 \cap A_2) . \]

Combined with the first result this gives equality.

2. Let \( A \) be the set of real numbers, and let a subset of \( A \) be called open if it is \( A \) or the null set or if it consists of points \( x \) such that \( x > k \) for some \( k \).

Prove that the open sets defined in this way form a topology for \( A \).

**Ans:** Denote the interval \((k, \infty)\) by \( S_k \). In this notation, \( A = S_{-\infty} \) and \( \phi = S_{\infty} \). Denote by \( \mathcal{T} \) this collection of sets.

We need to show that \( \mathcal{T} \) satisfies the axioms of a topology.
By construction, \( \phi \in \mathcal{T} \) and \( A \in \mathcal{T} \).
Consider the intersection of a finite number of sets in $T$. If $\phi$ is one of these, then the intersection is also $\phi$ which is an element of $T$.

Otherwise, let the sets be \{${S_{\alpha_1}, \ldots, S_{\alpha_n}}$\}.

If $\alpha_I$ is the maximum of \{${\alpha_1, \ldots, \alpha_n}$\}, then

$$I = \bigcap_{i=1}^n S_{\alpha_i} = S_{\alpha_I}$$

since, if $x \in I$, $x \in S_{\alpha_I}$, while if $x \in S_{\alpha_i}$, then $x > \alpha_I$, so that $x > \alpha_i$ for $i = 1 \ldots n$. Therefore $x \in S_{\alpha_i}$ for all $i$ and hence $x \in I$.

Consider the union of an arbitrary number of sets in $T$.

Let the sets be $S_{\alpha}$.

If the set of values \{${\alpha}$\} is not bounded below, then for any $x \in A$, there is $\alpha^* < x$, and $x \in S_{\alpha^*}$. Therefore the union of the sets is $A$ which is in $T$.

Otherwise, let $a = \inf\{\alpha\}$. If $x > a$, then $x$ is not a lower bound for $\{\alpha\}$, so that for some $\alpha^*$, $x > \alpha^* \geq a$.

Therefore

$$S_a \subset \bigcup_{\alpha} S_{\alpha}$$

On the other hand $\alpha \geq a$ for all $\alpha$, so that

$$S_{\alpha} \subset S_a : \bigcup_{\alpha} S_{\alpha} \subset S_a$$

Combining these results, we see that the union is $S_a \in T$.

Since $T$ satisfies the axioms, it is a topology for $A$.

3. If $M_1 = (A_1, d_1)$ and $M_2 = (A_2, d_2)$ are two metric spaces, show that the function defined by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

where $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$ is a metric on $A_1 \times A_2$.

Show that the topology generated by this metric is the product topology.

**Ans:** $d$ satisfies the axioms for a metric;

(a) $$d((y_1, y_2), (x_1, x - 2)) = d_1(y_1, x_1) + d_2(y_2, x_2)$$

$$= d_1(x_1, y_1) + d_2(x_2, y_2)$$

$$= d((x_1, x_2), (y_1, y_2))$$

(b) $$d_1(x_1, y_1) \geq 0 \ , \ d_2(x_2, y_2) \geq 0 \ ,$$

therefore $$d((x_1, x_2), (y_1, y_2)) \geq 0$$.

(c) $$d((x_1, x_2), (y_1, y_2)) = 0$$ iff $$d_1(x_1, y_1) = 0$$ and $$d_2(x_2, y_2) = 0$$

i.e. iff $x_1 = y_1$ and $x_2 = y_2$
(d) \[ d((x_1, x_2), (z_1, z_2)) = d_1(x_1, z_1) + d_2(x_2, z_2) \]
\[ \leq d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) \]
\[ = d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)) \]

Let \( U \) be open in \((A_1, d_1)\) and \( V \) be open in \((A_2, d_2)\). If \((x_0, y_0) \in U \times V\), then \(x_0 \in U\) and \(y_0 \in V\). Therefore there are positive constants \( \epsilon_1 \) and \( \epsilon_2 \) such that \( N_1 = \{ x \in A_1; d_1(x, x_0) < \epsilon_1 \} \subset U \) and \( N_2 = \{ y \in A_2; d_2(y, y_0) < \epsilon_2 \} \subset V \).

Then if \( \epsilon = \min(\epsilon_1, \epsilon_2) \), \( \{(x, y) \in A_1 \times A_2; d((x, y), (x_0, y_0)) < \epsilon \} \subset N_1 \times N_2 \subset U \times V \).

Therefore \( U \times V \) is open in \( A_1 \times A_2 \).

Since the product topology consists of unions of sets of this form, every set in the product topology is in the topology generated by \( d \).

Conversely, if \( W \) is in the topology generated by \( d \), then for each \((x_0, y_0) \in W\) there is \( \epsilon > 0 \) such that \[ N = \{(x, y) \in A_1 \times A_2; d((x, y), (x_0, y_0)) < \epsilon \} \subset W \]

Let \[ N_1 = \{ x \in A_1; d_1(x, x_0) < \frac{1}{2} \epsilon \} \subset U \]
and \[ N_2 = \{ y \in A_2; d_2(y, y_0) < \frac{1}{2} \epsilon \} \subset V \]

Then \( N_1 \times N_2 \) is in the product topology, and \[ N_1 \times N_2 \subset N \subset W \]

Since \[ W = \bigcup_{(x_0, y_0) \in W} N_1 \times N_2 \]
\( W \) is in the product topology. Therefore these two topologies are identical.
4. If $S, T$ are topological spaces homeomorphic respectively to $S', T'$, prove that $S \times T$ is homeomorphic to $S' \times T'$.

**Ans:** Since $S$ is homeomorphic to $S'$ there is a bi-continuous 1-1 map $f$ from $S$ to $S'$.

Similarly there is a bi-continuous 1-1 map $g$ from $T$ to $T'$.

Let $F$ be the map from $S \times T$ to $S' \times T'$ defined by $F(x, y) = (f(x), g(y))$.

If $F(x_1, y_1) = F(x_2, y_2)$, then $f(x_1) = f(x_2)$ and $g(y_1) = g(y_2)$. Since $f$ and $g$ are 1-1, $(x_1, y_1) = (x_2, y_2)$ and $F$ is also 1-1.

If $W'$ is open in $S' \times T'$, then $W'$ is the union of sets of the form $U' \times V'$, where $U'$ is open in $S'$ and $V'$ is open in $T'$.

Let $U = f^{-1}(U')$ and $V = g^{-1}(V')$. These are open sets in $S$ and $T$ respectively.

Then $F^{-1}(U' \times V') = U \times V$ is open in $S \times T$.

Therefore

$$F^{-1}(W') = F^{-1}\left(\bigcup U' \times V'\right)$$

$$= \bigcup (F^{-1}(U') \times V')$$

$$= \bigcup U \times V$$

is open in $S \times T$.

Therefore $F$ is continuous from $S \times T$ to $S' \times T'$.

Similarly $F^{-1}$ is continuous, so that $F$ is a homeomorphism from $S \times T$ to $S' \times T'$. 