MATH 3402

Tutorial Sheet 4

1. Show that the union of two compact sets is compact, and that the intersection of any number of compact sets is compact.

Ans. Any open cover of $X_1 \cup X_2$ is an open cover for X_1 and for X_2. Therefore there is a finite subcover for X_1 and a finite subcover for X_2. The union of these subcovers, which is finite, is a subcover for $X_1 \cup X_2$.

The intersection of any number of compact sets is a closed subset of any of the sets, and therefore compact.

2. List all possible topologies on

(i) $$\{a, b\}$$

(ii) $$\{a, b, c\}$$

Ans. (i)

$$\{\phi, X\}; \{\phi, \{a\}, X\}; \{\phi, \{b\}, X\}; 2^X$$

(ii)

$$\{\phi, X\}$$
$$\{\phi, \{a\}, X\}; \{\phi, \{b\}, X\}; \{\phi, \{c\}, X\}$$
$$\{\phi, \{b, c\}, X\}; \{\phi, \{a, c\}, X\}; \{\phi, \{a, b\}, X\}$$
$$\{\phi, \{a\}, \{b, c\}, X\}; \{\phi, \{b\}, \{a, c\}, X\}; \{\phi, \{c\}, \{a, b\}, X\}$$
$$\{\phi, \{a\}, \{a, b\}, X\}; \{\phi, \{b\}, \{b, c\}, X\}; \{\phi, \{c\}, \{a, c\}, X\}$$
$$\{\phi, \{b\}, \{a, b\}, X\}; \{\phi, \{c\}, \{b, c\}, X\}; \{\phi, \{a\}, \{a, c\}, X\}$$
$$\{\phi, \{a\}, \{b\}, \{a, b\}, X\}; \{\phi, \{b\}, \{c\}, \{b, c\}, X\}; \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$$
$$\{\phi, \{a\}, \{a, b\}, \{a, c\}X\}; \{\phi, \{b\}, \{b, c\}, \{a, b\}X\}; \{\phi, \{c\}, \{a, c\}, \{b, c\}X\}$$
$$2^X$$

3. Prove that any map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is continuous if either $\mathcal{T}_1$ is the discrete topology or $\mathcal{T}_2$ is the indiscrete topology.

Ans. If $\mathcal{T}_1$ is the discrete topology, then every set in 2^X is in $\mathcal{T}_1$. In particular $f^{-1}(U) \in \mathcal{T}_1$ for every $U \in \mathcal{T}_2$.

If $\mathcal{T}_2$ is the indiscrete topology, then $\mathcal{T}_2 = \{\phi, Y\}$.
$f^{-1}(Y) = X \in \mathcal{T}_1$, $f^{-1}(\phi) = \phi \in \mathcal{T}_1$.

4. Let (X, d) be $\mathbb{Q}$ with the usual metric.
Show that the set $S = \{x \in \mathbb{Q}; x^2 < 2\}$ is both open and closed in (X, d).

Ans. For $x \in S$, let $\epsilon = \sqrt{2} - |x|$. Then $\mathcal{N}(x, \epsilon) \subset S$ so that S is open.
For $x \in \backslash s$, let $\epsilon = |x| - \sqrt{2}$. Then $\mathcal{N}(x, \epsilon) \subset \backslash S$ so that $\backslash S$ is open ans S is closed.

5. Prove that $f : (X, \mathcal{T}) \to \mathbb{R}$ is continuous if and only if for every $a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ and $f^{-1}((a, \infty))$ are in $\mathcal{T}$.

Ans. For any $a < b$, $f^{-1}((-\infty, b)) \in \mathcal{T}$ and $f^{-1}((a, \infty)) \in \mathcal{T}$. Therefore

$$f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty)) \in \mathcal{T}$$
$$f^{-1}((-\infty, b) \cap (a, \infty)) = f^{-1}((a, b)) \in \mathcal{T}$$

Since every open set U in $\mathbb{R}$ is the union of a countable collection of disjoint intervals,

$$f^{-1}(U) = f^{-1}(\cup(a_i, b_i)) = \cup f^{-1}((a_i, b_i)) \in \mathcal{T}$$

and f is continuous.