1. Show that the union of two compact sets is compact, and that the intersection of any number of compact sets is compact.

**Ans.** Any open cover of $X_1 \cup X_2$ is an open cover for $X_1$ and for $X_2$. Therefore there is a finite subcover for $X_1$ and a finite subcover for $X_2$. The union of these subcovers, which is finite, is a subcover for $X_1 \cup X_2$.

The intersection of any number of compact sets is a closed subset of any of the sets, and therefore compact.

2. List all possible topologies on

(i) $\{a, b\}$

(ii) $\{a, b, c\}$

**Ans.** (i) $\{\emptyset, X\}; \{\emptyset, \{a\}, X\}; \{\emptyset, \{b\}, X\}; \{\emptyset, \{b\}, X\}; 2^X$

(ii) $\{\emptyset, X\}$

$\{\emptyset, \{a\}, X\}; \{\emptyset, \{b\}, X\}; \{\emptyset, \{c\}, X\}$

$\{\emptyset, \{a\}, \{b\}, X\}; \{\emptyset, \{a\}, \{c\}, X\}; \{\emptyset, \{a\}, \{b\}, X\}$

$\{\emptyset, \{a\}, \{b\}, \{c\}, X\}; \{\emptyset, \{a\}, \{b\}, \{c\}, X\}; \{\emptyset, \{a\}, \{b\}, \{c\}, X\}$

$\{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}; \{\emptyset, \{b\}, \{a,b\}, X\}; \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$

$2^X$

3. Prove that any map $f : (X, T_1) \rightarrow (Y, T_2)$ is continuous if either $T_1$ is the discrete topology or $T_2$ is the indiscrete topology.

**Ans.** If $T_1$ is the discrete topology, then every set in $2^X$ is in $T_1$. In particular $f^{-1}(U) \in T_1$ for every $U \in T_2$.

If $T_2$ is the indiscrete topology, then $T_2 = \{\emptyset, Y\}$.

$f^{-1}(Y) = X \in T_1$, $f^{-1}(\emptyset) = \emptyset \in T_1$.

4. Let $(X, d)$ be $\mathbb{Q}$ with the usual metric.

Show that the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$ is both open and closed in $(X, d)$.

**Ans.** For $x \in S$, let $\epsilon = \sqrt{2} - |x|$. Then $N(x, \epsilon) \subset S$ so that $S$ is open.

For $x \in \mathbb{Q} \setminus S$, let $\epsilon = |x| - \sqrt{2}$. Then $N(x, \epsilon) \subset \mathbb{Q} \setminus S$ so that $\mathbb{Q} \setminus S$ is open and $S$ is closed.
5. Prove that \( f : (X, \mathcal{T}) \rightarrow \mathbb{R} \) is continuous if and only if for every \( a \in \mathbb{R} \), \( f^{-1}((-\infty, a)) \) and \( f^{-1}((a, \infty)) \) are in \( \mathcal{T} \).

**Ans.** For any \( a < b \), \( f^{-1}((-\infty, b)) \in \mathcal{T} \) and \( f^{-1}((a, \infty)) \in \mathcal{T} \). Therefore

\[
\begin{align*}
f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty)) & \in \mathcal{T} \\
f^{-1}((-\infty, b) \cap (a, \infty)) & = f^{-1}((a, b)) \in \mathcal{T}
\end{align*}
\]

Since every open set \( U \) in \( \mathbb{R} \) is the union of a countable collection of disjoint intervals,

\[
f^{-1}(U) = \bigcup f^{-1}((a_i, b_i)) = \bigcup f^{-1}((a_i, b_i)) \in \mathcal{T}
\]
and \( f \) is continuous.