MATH 3402

Tutorial Sheet 3

Solutions

1. Show that a finite union of bounded sets is bounded.

Ans. Consider

$$\bigcup_{i=1}^{n} S_i$$

where each S_i is bounded in (X, d).

Therefore, for some fixed $a \in X$ and for each i there is a constant k_i such that $d(x, a) \leq k_i$ for each $x \in S_i$.

Let $k = \max_i k_i$.

If $x \in \cup S_i$, $x \in S_i$ for some i. Then, $d(x, a) \leq k_i \leq k$, and $\cup S_i$ is bounded.

Show that the intersection of an arbitrary number of bounded sets is either bounded or empty.

Ans. Let S_α be a collection of bounded sets in (X, d).

For some fixed $a \in X$ and for each α there is a constant $k_\alpha \geq 0$ such that $d(x, a) \leq k_\alpha$ for all $x \in S_\alpha$.

Let $k = \inf\{k_\alpha\}$. If $x \in S_\alpha$ for every α, then $d(x, a) \leq k_\alpha$ for every α, so that $d(x, a) \leq k$ and $\cap S_\alpha$ is bounded if it is not empty.

2. Show that if a metric space (X, d) has the property that every bounded sequence converges, then X consists of only one point.

Ans. Since the sequences converge, they have a limit in X, so that X is not empty.

If x_1 and x_2 are distinct points of X, then the sequence

$$a_n = \begin{cases} x_1 \text{ if } n = 2m \\ x_2 \text{ if } n = 2m + 1 \end{cases}$$

is a bounded sequence in (X, d) which does not converge.

3. Say whether the following sequences converge, and find their limit if they do:

a) $a_n = x^{-n}$ in $C(\frac{1}{3}, \frac{2}{3})$ (the continuous functions on the open interval $(\frac{1}{3}, \frac{2}{3})$) with the uniform - sup - metric;

Ans. Since $||a_n|| = 3^n$, the sequence is not bounded and therefore does not converge.

b) $a_n = e^{-nx}$ in $C[0, 1]$ with the uniform metric.

Ans. The limit function is discontinuous, therefore this sequence does not converge.

c) $a_n = (\alpha_n, f(\alpha_n))$ in $\mathbb{R}$ with the Euclidean metric, where α_n is a convergent sequence in $\mathbb{R}$ with limit α, and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Given any $\epsilon > 0$, there is $\delta > 0$ such that $|x - \alpha| < \delta$ implies $|f(x) - f(\alpha)|$.

Let $\delta^* = min(\epsilon, \delta)$.

Since $\{\alpha_n\}$ converges to α, there is N such that $|\alpha_n - \alpha| < \delta^*$ for all $n > N$.

Therfore $|f(\alpha_n) - f(\alpha)| < \epsilon$ for all $n > N$, and

$$\left((\alpha_n - \alpha)^2 + (f(\alpha_n) - f(\alpha))^2\right)^{1/2} < \epsilon\sqrt{2}$$

for all $n > N$.

4. Determine the diameter of the set $\{|z| < r\}$ in $(\mathbb{C}, d)$ where

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

Ans. The circle $|z| = r$ in the complex plane maps onto a circle of latitude on the Riemann Sphere under the stereographic projection.

The point $z = r$ maps onto the point $(2r/(1+r^2), 0, (r^2-1)/(r^2+1))$, so that the diameter of this circle is $4r/(1+r^2)$.

If $r \leq 1$, this is the diameter of the set $\{|z| < r\}$.

If $r > 1$, the set includes the whole of the southern hemisphere, and the diameter of the set is 2.

Repeat the exercise for the set $\{|z-1| < r\}$.

Ans. When the circle $|z-1| = r$ is mapped onto the Riemann sphere, the most widely separated points are the images of $1+r$ and $1-r$ which map onto

$$\left(\frac{2r+2}{r^2+2r^2+2}, 0, \frac{r^2+2r}{r^2+2r+2}\right) \quad \text{and} \quad \left(\frac{2-2r}{r^2-2r+2}, 0, \frac{r^2-2r}{r^2-2r+2}\right)$$

The distance between these points is $4r/\sqrt{r^4+4}$, which is the diameter of the set for $r \leq \sqrt{2}$.

For $r > \sqrt{2}$, the diameter of the set is 2.

5. Let $\{f_n(x)\}$ be a sequence of functions, continuous on the closed interval $[a,b] \in \mathbb{R}$, which is a Cauchy sequence with respect to the uniform metric on $C[a,b]$.

a) Prove that the sequence converges pointwise on $[a,b]$.

Ans. Since $\{f_n\}$ is a Cauchy sequence with respect to the sup metric on $[a,b]$, given any $\epsilon > 0$ there is N such that

$$\sup_{a \leq x \leq b} |f_n(x) - f_m(x)| < \epsilon \ \forall\ m, n > N \ .$$

Therefore for each $x_0 \in [a,b]$,

$$|f_n(x_0) - f_m(x_0)| < \epsilon \ \forall\ m, n > N \ ,$$

and $\{f_n(x_0)\}$ is a Cauchy sequence in $\mathbb{R}$, which converges in $\mathbb{R}$.

b) Prove that the function defined by these pointwise limits is continuous on $[a,b]$.

Ans. Let $f(x)$ denote the pointwise limit of the sequence.

Given any $\epsilon > 0$, there is N such that

$$\sup_{a \leq x \leq b} |f_n(x) - f(x)| < \frac{1}{3}\epsilon \ \forall\ n > N \ .$$

Choose $n_1 > N$. Since f_{n_1} is continuous on $[a,b]$, for every x_0 in $[a,b]$ there exists $\delta > 0$ such that

$$|f_{n_1}(x) - f_{n_1}(x_0)| < \frac{1}{3}\epsilon \ \forall\ x \in [a,b]; |x - x_0| < \delta \ .$$

Therefore

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_1}(x)| + |f_{n_1}(x) - f_{n_1}(x_0)| + |f_{n_1}(x_0) - f(x_0)| < \epsilon$$

for all $x \in [a,b]; |x - x_0| < \delta$.

6. Show that the taxicab and Euclidean metrics are topologically equivalent on $\mathbb{R}^2$.

Ans.

$$(|x_1 - x_2| + |y_1 - y_2|)^2 = (x_1 - x_2)^2 + 2|x_1 - x_2|\,|y_1 - y_2| + (y_1 - y_2)^2$$
$$2|x_1 - x_2|\,|y_1 - y_2| \le (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$d_2((x_1, y_1), (x_2, y_2))^2 \le d_1((x_1, y_1), (x_2, y_2))^2 \le 2d_2((x_1, y_1), (x_2, y_2))^2$$
$$d_2 \le d_1 \le \sqrt{2}d_2$$

7. Show that the metrics

$$d_1(z_1, z_2) = |z_1 - z_2|$$

and

$$d_2 = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}$$

are topologically equivalent on $\mathbb{C}$.

Ans. $d_2(z_1, z_2) \le 2d_1(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{C}$, therefore

$$\{d_1(z, a) < \frac{1}{2}\epsilon\} \subset \{d_2(z, a) < \epsilon\}$$

and any set open in $(\mathbb{C}, d_2)$ is open in $(\mathbb{C}, d_1)$.

If $|z - a| < \epsilon$,

$$\begin{aligned} d_1(z, a) =& \frac{1}{2}\sqrt{1 + |a|^2}\sqrt{1 + |z|^2} d_2(z, a) \\ \le& \frac{1}{2}\sqrt{1 + |a|^2}\sqrt{1 + (|a| + \epsilon)^2} d_2(z, a) \end{aligned}$$

Let

$$\epsilon_1 = \frac{2\epsilon}{\sqrt{1 + |a|^2}\sqrt{1 + (|a| + \epsilon)^2}}$$

then

$$\{d_2(z, a) < \epsilon_1\} \subset \{d_1(z, a) < \epsilon\}$$

and any set open in $(\mathbb{C}, d_1)$ is open in $(\mathbb{C}, d_2)$.