1. In each of the following cases, say whether \((X,d)\) is a metric space or not. If it is not, say which of the axioms fails.

(a) \(X = \mathbb{R}^2\); \(d((x,y),(x',y')) = |y - y'|\).

**Ans.** \(d((x,0),(x',0)) = 0\) therefore axiom 2 fails.

(b) \(X = \mathbb{Q}\); \(d(x,y) = (x-y)^3\).

**Ans.** This is the standard metric on \(\mathbb{C}\).

(c) \(X = \mathbb{C}\); \(d(x,y) = (x-y)^3\).

**Ans.** \(d(y,x) = -d(x,y)\). Axioms 2 and 3 fail.

(d) \(X = \mathbb{C}\); \(d(z_1,z_2) = \min\{|z_1| + |z_2|, |z_1 - 1| + |z_2 - 1|\}\) if \(z_1 \neq z_2\), \(d(z,z) = 0\).

**Ans.** The first three axioms are obviously satisfied.

If either \(z_1 = z_2\) or \(z_2 = z_3\) or \(z_1 = z_3\), the fourth axiom is trivially satisfied.

Therefore assume that all the points are distinct.

The metric is invariant under the mapping \(z : \rightarrow 1 - z\), therefore, wlog we can take \(d(z_1,z_3) = |z_1| + |z_3|\), and consider the possible values of \(d(z_1,z_2)\) and \(d(z_2,z_3)\).

If \(d(z_1,z_2) = |z_1| + |z_2|\) and \(d(z_2,z_3) = |z_2| + |z_3|\), then

\[
d(z_1,z_2) + d(z_2,z_3) = |z_1| + |z_3| + 2|z_2| \geq d(z_1,z_3).
\]

If \(d(z_1,z_2) = |z_1| + |z_2|\) and \(d(z_2,z_3) = |z_2 - 1| + |z_3 - 1|\), then

\[
d(z_1,z_3) = |z_1| + |z_3|
\]

\[= |z_1| + |(z_3 - 1) - (z_2 - 1) + z_2| \leq |z_1| + |z_3 - 1| + |z_2 - 1| + |z_2| \]

\[\leq d(z_1,z_2) + d(z_2,z_3)\]

If \(d(z_1,z_2) = |z_1 - 1| + |z_2 - 1|\) and \(d(z_2,z_3) = |z_2 - 1| + |z_3 - 1|\), then

\[
d(z_1,z_2) + d(z_2,z_3) = |z_1 - 1| + |z_3 - 1| + 2|z_2 - 1|
\]

\[\geq |z_1 - 1| + |z_3 - 1| \geq d(z_1,z_3).
\]

Therefore \((\mathbb{C},d)\) is a metric space.

(e) \(X = \mathbb{R}\); \(d(x,y) = \left| \int_x^y f(t) \, dt \right|\).

where \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a given positive integrable function.
Ans. Again the first three axioms are obviously satisfied. To show that the fourth axiom is satisfied, assume wlog that $x < z$ so that

$$d(x, z) = \int_x^z f(t) \, dt.$$  

If $x < y < z$,

$$d(x, z) = \int_x^z f(t) \, dt = \int_x^y f(t) \, dt + \int_y^z f(t) \, dt = d(x, y) + d(y, z).$$

If $x < z < y$,

$$d(x, y) = d(x, z) + d(z, y); d(x, z) \leq d(x, y) \leq d(y, x) + d(y, z)$$

If $y < x < z$,

$$d(y, z) = d(y, x) + d(x, z); d(x, z) \leq d(y, z) \leq d(x, y) + d(y, z).$$

Therefore $(\mathbb{R}, d)$ is a metric space.

2. Sketch the sets $\{x \in \mathbb{R}^2, d(x, 0) < 1\}$ when $d(x, y)$ is

(a) The Euclidean metric;
Ans. The interior of the unit circle.

(b) The taxicab metric;
Ans. The interior of the square with vertices $(\pm 1, 0), (0, \pm 1)$.

(c) The sup metric;
Ans. The interior of the square with vertices $(\pm 1, \pm 1)$.

(d) The discrete metric.
Ans. The point $x = 0$.

3. For $z_j = x_j + iy_j \in \mathbb{C}$, let

$$\xi_j = \frac{2x_j}{1 + |z_j|^2}; \eta_j = \frac{2y_j}{1 + |z_j|^2}; \zeta_j = \frac{1 - |z_j|^2}{1 + |z_j|^2}.$$  

Show that

(i) \[ \xi_j^2 + \eta_j^2 + \zeta_j^2 = 1 \]

Ans.

$$\frac{\xi_j^2 + \eta_j^2 + \zeta_j^2}{1 + 2|z_j|^2 + |z_j|^4} = 1$$

(ii) \[ ((\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2)^{1/2} = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}} \]
\[(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2\]
\[= \xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 + \eta_1^2 - 2\eta_1\eta_2 + \eta_2^2 + \zeta_1^2 - 2\zeta_1\zeta_2 + \zeta_2^2\]
\[= 2 - 2(\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2)\]
\[= 2 - \frac{8x_1x_2 + 8y_1y_2 + 2 - 2|z_1|^2 - 2|z_2|^2 + 2|z_1|^2|z_2|^2}{1 + |z_1|^2 + |z_2|^2 + |z_1|^2|z_2|^2}\]
\[= 4\left(\frac{|z_1|^2 - 2x_1x_2 - 2y_1y_2 + |z_2|^2}{(1 + |z_1|^2)(1 + |z_2|^2)}\right)\]
\[= 4\left(\frac{x_1^2 + y_1^2 - 2x_1x_2 - 2y_1y_2 + x_2^2 + y_2^2}{(1 + |z_1|^2)(1 + |z_2|^2)}\right)\]
\[= 4\left(\frac{|z_1 - z_2|^2}{(1 + |z_1|^2)(1 + |z_2|^2)}\right)\]

Hence deduce that
\[d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}\]
is a metric on \(\mathbb{C}\).

**Ans.** This function is derived from the Euclidean metric in \(\mathbb{R}^3\) (restricted to the unit sphere), therefore the axioms for a metric satisfied by \(d\).

Show that the sequence \(\{z_n = n\}\) is a Cauchy sequence with respect to this metric.

**Ans.** Suppose wlog that \(m > n\).

\[d(m, n) = \frac{2(m - n)}{\sqrt{1 + m^2\sqrt{1 + n^2}}} < \frac{2(1 - (n/m))}{\sqrt{1 + n^2}} < \frac{2}{n}\]

Given any \(\epsilon > 0\), choose \(N = \lceil 2/\epsilon \rceil\).

For \(m > n > N\), \(d(m, n) < \frac{2}{n} < \epsilon\) as required.

4. Show that if for some \(x_0 \in S\), \(d(x, x_0) < k\) for all \(x \in Q\), then for any \(a \in S\) there is a constant \(k_a\) such that \(d(x, a) < k_a\) for all \(x \in Q\).

(That is, a set \(Q\) is bounded in \(S\) if it is bounded with respect to some member of \(S\).)

**Ans.** For any \(a \in S\) and any \(x \in Q \subset S\),

\[d(x, a) \leq d(x, x_0) + d(x_0, a) < k + d(x_0, a) = k_a\]