MATH 3402
TUTORIAL SHEET 2
SOLUTIONS

1. In each of the following cases, say whether (X, d) is a metric space or not. If it is not, say which of the axioms fails.

(a)
$X = \mathbb{R}^2$; $d((x,y),(x',y')) = |y - y'|$.

Ans. $d((x,0),(x',0)) = 0$ therefore axiom 2 fails

(b)
$X = \mathbb{C}$; $d(z_1, z_2) = |z_1 - z_2|$.

Ans. This is the standard metric on $\mathbb{C}$.

(c)
$X = \mathbb{Q}$; $d(x,y) = (x-y)^3$.

Ans. $d(y,x) = -d(x,y)$. Axioms 2 and 3 fail.

(d)
$X = \mathbb{C}$; $d(z_1, z_2) = \min\{|z_1| + |z_2|, |z_1 - 1| + |z_2 - 1|\}$ if $z_1 \neq z_2$, $d(z,z) = 0$.

Ans. The first three axioms are obviously satisfied.

If either $z_1 = z_2$ or $z_2 = z_3$ or $z_1 = z_3$, the fourth axiom is trivially satisfied. Therefore assume that all the points are distinct.

The metric is invariant under the mapping $z :\to 1 - z$, therefore, wlog we can take $d(z_1, z_3) = |z_1| + |z_3|$, and consider the possible values of $d(z_1, z_2)$ and $d(z_2, z_3)$.

If $d(z_1, z_2) = |z_1| + |z_2|$ and $d(z_2, z_3) = |z_2| + |z_3|$, then

$$d(z_1, z_2) + d(z_2, z_3) = |z_1| + |z_3| + 2|z_2| \geq d(z_1, z_3) \ .$$

If $d(z_1, z_2) = |z_1| + |z_2|$ and $d(z_2, z_3) = |z_2 - 1| + |z_3 - 1|$, then

$$\begin{aligned} d(z_1, z_3) &= |z_1| + |z_3| \\ &= |z_1| + |(z_3 - 1) - (z_2 - 1) + z_2| \leq |z_1| + |z_3 - 1| + |z_2 - 1| + |z_2| \\ &\leq d(z_1, z_2) + d(z_2, z_3) \end{aligned}$$

If $d(z_1, z_2) = |z_1 - 1| + |z_2 - 1|$ and $d(z_2, z_3) = |z_2 - 1| + |z_3 - 1|$, then

$$\begin{aligned} d(z_1, z_2) + d(z_2, z_3) &= |z_1 - 1| + |z_3 - 1| + 2|z_2 - 1| \\ &\geq |z_1 - 1| + |z_3 - 1| \geq d(z_1, z_3) \ . \end{aligned}$$

Therefore $(\mathbb{C}, d)$ is a metric space.

(e)
$$X = \mathbb{R} \ ; \ d(x,y) = \left| \int_x^y f(t)\, dt \right| \ ,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given positive integrable function.

Ans. Again the first three axioms are obviously satisfied.
To show that the fourth axiom is satisfied, assume wlog that $x < z$ so that

$$d(x,z) = \int_x^z f(t)\,dt\ .$$

If $x < y < z$,

$$d(x,z) = \int_x^z f(t)\,dt = \int_x^y f(t)\,dt + \int_y^z f(t)\,dt = d(x,y) + d(y,z)\ .$$

If $x < z < y$,

$$d(x,y) = d(x,z) + d(z,y); d(x,z) \leq d(x,y) \leq d(x,y) + d(y,z)$$

If $y < x < z$,

$$d(y,z) = d(y,x) + d(x,z); d(x,z) \leq d(y,z) \leq d(x,y) + d(y,z)$$

Therefore $(\mathbb{R}, d)$ is a metric space.

2. Sketch the sets $\{\underline{x} \in \mathbb{R}^2, d(\underline{x}, \underline{0}) < 1\}$ when $d(\underline{x}, \underline{y})$ is
(a) The Euclidean metric;

Ans. The interior of the unit circle.

(b) The taxicab metric;

Ans. The interior of the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$.

(c) The sup metric;

Ans. The interior of the square with vertices $(\pm 1, \pm 1)$.

(d) The discrete metric.

Ans. The point $\underline{x} = \underline{0}$.

3. For $z_j = x_j + iy_j \in \mathbb{C}$, let

$$\xi_j = \frac{2x_j}{1 + |z_j|^2}\ ;\ \eta_j = \frac{2y_j}{1 + |z_j|^2}\ ;\ \zeta_j = \frac{1 - |z_j|^2}{1 + |z_j|^2}\ .$$

Show that

(i)
$$\xi_j^2 + \eta_j^2 + \zeta_j^2 = 1$$

Ans.

$$\xi_j^2 + \eta_j^2 + \zeta_j^2$$
$$= \frac{4x_j^2 + 4y_j^2 + 1 - 2|z_j|^2 + |z_j|^4}{1 + 2|z_j|^2 + |z_j|^4} = 1$$

(ii)
$$\left((\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2\right)^{1/2} = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2}\sqrt{(1 + |z_2|^2)}}$$

Ans.

$$\begin{aligned}
&(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2\\
&=\xi_1^2-2\xi_1\xi_2+\xi_2^2+\eta_1^2-2\eta_1\eta_2+\eta_2^2+\zeta_1^2-2\zeta_1\zeta_2+\zeta_2^2\\
&=2-2(\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2)\\
&=2-\frac{8x_1x_2+8y_1y_2+2-2|z_1|^2-2|z_2|^2+2|z_1|^2|z_2|^2}{1+|z_1|^2+|z_2|^2+|z_1|^2|z_2|^2}\\
&=4\frac{|z_1|^2-2x_1x_2-2y_1y_2+|z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}\\
&=4\frac{x_1^2+y_1^2-2x_1x_2-2y_1y_2+x_2^2+y_2^2}{(1+|z_1|^2)(1+|z_2|^2)}\\
&=4\frac{(x_1-x_2)^2+(y_1-y_2)^2}{(1+|z_1|^2)(1+|z_2|^2)}\\
&=4\frac{|z_1-z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}
\end{aligned}$$

Hence deduce that

$$d(z_1,z_2)=\frac{2|z_1-z_2|}{\sqrt{(1+|z_1|^2}\sqrt{(1+|z_2|^2)}}$$

is a metric on $\mathbb{C}$.

Ans. This function is derived from the Euclidean metric in $\mathbb{R}^3$ (restricted to the unit sphere), therefore the axioms for a metric satisfied by d.

Show that the sequence $\{z_n = n\}$ is a Cauchy sequence with respect to this metric.

Ans. Suppose wlog that $m > n$.

$$\begin{aligned}
d(m,n) &= \frac{2(m-n)}{\sqrt{1+m^2}\sqrt{1+n^2}}\\
&< \frac{2(1-(n/m)}{\sqrt{1+n^2}}\\
&<\frac{2}{n}
\end{aligned}$$

Given any $\epsilon > 0$, choose $N = [2/\epsilon]$.
For $m > n > N$, $d(m,n) < \frac{2}{n} < \epsilon$ as required.

4. Show that if for some $x_0 \in S$, $d(x,x_0) < k$ for all $x \in Q$, then for any $a \in S$ there is a constant k_a such that $d(x,a) < k_a$ for all $x \in Q$.

(That is, a set Q is bounded in S if it is bounded with respect to some member of S.)

Ans. For any $a \in S$ and any $x \in Q \subset S$,

$$d(x,a) \le d(x,x_0) + d(x_0,a) < k + d(x_0,a) = k_a\ .$$