**MATH 3402**
TUTORIAL SHEET 1
SOLUTIONS

1.Describe each of the following sets as the empty set, as $\mathbb{R}$, or in interval notation as appropriate:

(a) $$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$$

(b) $$\bigcup_{n=1}^{\infty}(-n,n)$$

(c) $$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},1+\frac{1}{n}\right)$$

(d) $$\bigcup_{n=1}^{\infty}\left(-\frac{1}{n},2+\frac{1}{n}\right)$$

(e) $$\bigcup_{n=1}^{\infty}\left(\mathbb{R}\backslash\left(-\frac{1}{n},\frac{1}{n}\right)\right)$$

(f) $$\bigcap_{n=1}^{\infty}\left(\mathbb{R}\backslash\left[\frac{1}{n},2+\frac{1}{n}\right]\right)$$

**Ans.**

(a) $-\frac{1}{n} < 0 < \frac{1}{n}\ \forall\ n$, therefore $0 \in \cap(-\frac{1}{n},\frac{1}{n})$;
If $x > 0, \exists N; x > \frac{1}{N}$, therefore $x \notin (-\frac{1}{N},\frac{1}{N})$, therefore $x \notin \cap(-\frac{1}{n},\frac{1}{n})$.
Similarly, if $x < 0, x \notin \cup(-\frac{1}{n},\frac{1}{n})$, therefore $\cap(-\frac{1}{n},\frac{1}{n}) = \{0\} = [0,0]$.

(b) $\forall\ x \in \mathbb{R}, \exists N; N \leq x < N+1$, therefore if $x \geq 0, x \in (-N-1,N+1)$ and if $x < 0, x \in (N-1,1-N)$, therefore, $\cup(-n,n) = \mathbb{R}$.

(c) Similar to (a). $\cap(-\frac{1}{n},1+\frac{1}{n}) = \{0 \leq x \leq 1\} = [0,1]$.

(d) Since $(-\frac{1}{n},2+\frac{1}{n}) \subset (-1,3)\ \forall\ n, \cup(-\frac{1}{n},2+\frac{1}{n}) = (-1,3)$.

(e) $\cup(\mathbb{R}\backslash(-\frac{1}{n},\frac{1}{n}) = \mathbb{R}\backslash(\cap(-\frac{1}{n},\frac{1}{n})) = \mathbb{R}\backslash\{0\} = (-\infty,0)\cup(0,\infty)$.

(f) $\cup(\mathbb{R}\backslash[\frac{1}{n},2+\frac{1}{n}]) = \mathbb{R}\backslash(\cup[\frac{1}{n},2+\frac{1}{n}]) = \mathbb{R}\backslash\,(0,3] = (-\infty,0]\cup(3,\infty)$.

2. Show that if $A \subset B \subset \mathbb{R}$, and if $B$ is bounded above, then $A$ is bounded above, and $\sup A \leq \sup B$.

**Ans.**

$$\exists b\ ;\ x \in B \Rightarrow\ x \leq b\ .$$

But, if $x \in A$, $x \in B$, so that $x \leq b$ and $A$ is bounded above.

Since any upper bound for $B$ is an upper bound for $A$, in particular $\sup B$ is an upper bound for $A$.

Hence $\sup A \leq \sup B$.

3. Let $a_0$ and $a_1$ be distinct real numbers.
Define $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for each positive integer $n \geq 2$.
Show that $\{a_n\}$ is a Cauchy sequence.

**Ans.**

$$\begin{aligned} a_n - a_{n-1} &= \frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1} \\ &= -\frac{1}{2}(a_{n-1} - a_{n-2}) \\ &= \left(-\frac{1}{2}\right)^{n-1} (a_1 - a_0) \end{aligned}$$

Therefore, for $n > m$,

$$\begin{aligned} a_n - a_m &= \sum_{r=m+1}^{n} (a_r - a_{r-1}) \\ &= \sum_{r=m+1}^{n} \left(-\frac{1}{2}\right)^{r-1} (a_1 - a_0) \\ &= \frac{\left(-\frac{1}{2}\right)^{m-1} - \left(-\frac{1}{2}\right)^{n}}{1 - \left(-\frac{1}{2}\right)} (a_1 - a_0) \\ |a_n - a_m| &\leq \frac{2}{3} 2 \left(\frac{1}{2}\right)^{m-1} |a_1 - a_0| \\ &= \frac{8}{3} |a_1 - a_0| \, 2^{-m} \end{aligned}$$

Given $\epsilon > 0$, determine $N$ such that

$$\frac{8}{3} |a_1 - a_0| \, 2^{-N} < \epsilon$$

and then

$$|a_n - a_m| < \epsilon \; \forall \; n > m > N \; .$$

4. Suppose $x$ is an accumulation point of $\{a_n : n \in \mathbb{N}\}$.
Show that there is a subsequence of $\{a_n\}$ that converges to $x$.

**Ans.**

Since $x$ is an accumulation point, given any $\epsilon > 0$ there are infinitely many elements of $a_i \in \{a_n\}$ such that $|a_i - x| < \epsilon$.

In particular, for $\epsilon = 1$ we can find $a_{i_1}$ such that $|a_{i_1} - x| < 1$ .

Now for $\epsilon < \frac{1}{2}$ there are infinitely many $a_i$ such that $|a_i - x| < \frac{1}{2}$ . In particular, we can find an $a_{i_2}$ in this collection for which $i_2 > i_1$.

Similarly, with $\epsilon = \frac{1}{4}$ we can find $a_{i_3}$ with $i_3 > i_2 > i_1$ and $|a_{i_3} - x| < \frac{1}{4}$ .

Proceeding in this fashion we construct the required subsequence.

5. Given the non-negative real numbers $a_1, a_2, \ldots, a_r$, let $a = \sup\{a_i\}$. Prove that for any integer $n$,

$$a^n \leq a_i^n + a_2^n + \cdots + a_r^n \leq r a^n \ ,$$

and determine

$$\lim n \to \infty \, (a_1^n + a_2^n + \cdots + a_r^n)^{1/n} \ \ .$$

**Ans.**

Since the set $\{a_i\}$ is finite, $a = a_i$ for some $i$.

$$\text{Therefore} \quad a^n \leq a_1^n + a_2^n + \cdots + a_r^n \ .$$

Since $a = \sup\{a_i\}$, $a \geq a_i$ for each $i$.

$$\text{Therefore} \quad a_1^n + a_2^n + \cdots + a_r^n \leq r a^n \ .$$

Taking the $n^{th}$ root, we obtain

$$a \leq (a_1^n + a_2^n + \cdots + a_r^n)^{1/n} \leq r^{1/n} a \ .$$

As $n \to \infty$, $r^{1/n} \to 1$, so that

$$\lim n \to \infty \, (a_1^n + a_2^n + \cdots + a_r^n)^{1/n} \ = a \ .$$

6. Consider the sequence

$$0, \ 1, \ -1, \ 2, \ -2, \ \frac{1}{2}, \ -\frac{1}{2}. \ \ldots$$

used to demonstrate the countability of the rationals.

a) What is the fiftieth term in this sequence?

b) Which term in the sequence is $-\frac{4}{5}$ ?

**Ans.**

The fiftieth term in this sequence will be the twentyfifth term in the corresponding sequence of positive rationals:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ | 3 | 4 | $\frac{3}{2}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ |
| 5 | 6 | $\frac{5}{2}$ | $\frac{4}{3}$ | $\frac{3}{4}$ | $\frac{2}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{3}{5}$ | $\frac{5}{3}$ |
| 7 | 8 | $\frac{7}{2}$ | $\frac{5}{4}$ | $\frac{4}{5}$ | $\frac{2}{7}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $\frac{3}{7}$ | $\frac{7}{3}$ |

namely $\frac{4}{5}$.

$-\frac{4}{5}$ is the fiftyfirst term in the sequence.