ASSIGNMENT 2 FOR MATH3402 IN
2008 (DUE DATE: 24 APRIL 2008)

PLEASE SUBMIT IT TO THE ASSIGNMENT BOX,
LEVEL THREE, PRIESTLEY BUILDING #67

Question 1. (3 marks) Let $f : X \to Y$ be a function. Let $A$ and $B$ are two subsets of $Y$. Show

(i) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
(ii) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

Question 2. (4 marks)

(i) Let $(S, d)$ be a metric space (i.e. $d$ is a metric on $S$). Show that for any $x, y, z \in S$
$$|d(x, z) - d(y, z)| \leq d(x, y)$$

(ii) Let $(S, d)$ be a metric space. Take $a \in S$ and let $f : S \to \mathbb{R}$ be a function defined by
$$f(x) = d(x, a), \quad \forall x \in S.$$ Then show that $f$ is continuous at each $x_0 \in S$.

Question 3. (4 marks)

Let $(X, d_X)$ be a metric space and $(Y, d_Y)$ be another metric space. Define
$$\tilde{d}_X(x, y) = \frac{d_X(x, y)}{1 + d_X(x, y)} \quad \forall x, y \in X.$$ $(X, \tilde{d}_X)$ is also a metric space. Prove that $f : (X, \tilde{d}_X) \to (Y, d_Y)$ is continuous if and only if $f : (X, d_X) \to (Y, d_Y)$ is continuous

Question 4 is in the next page
**Question 4.** (4 marks + 1* bonus mark)

Let $p > 1$, $q > 1$ be the dual indices, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and let $X = C^0([a, b]; \mathbb{R})$ be the space of all continuous functions on $[a, b]$ with two real numbers $a < b$. Assume $f(x)$ and $g(x)$ be continuous functions on $[a, b]$ i.e. $f, g \in C^0([a, b]; \mathbb{R})$. Then

(i) Use Young’s inequality to prove

$$
\int_a^b |f(x)| |g(x)| \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \left( \int_a^b |g(x)|^q \, dx \right)^{1/q}
$$

where (4.1) is called “Hölder’s inequality”.

(ii) Use the inequality (4.1) to prove

$$
\left( \int_a^b |f(x) + g(x)|^p \, dx \right)^{1/p} \leq \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} + \left( \int_a^b |g(x)|^p \, dx \right)^{1/p}
$$

(iii) Use the inequality (4.2) to prove that

$$
d(f, g) = \|f - g\|_p = \left( \int_a^b |f(x) - g(x)|^p \, dx \right)^{1/p}
$$

is a metric on the space $C^0([a, b]; \mathbb{R})$ of all continuous functions on $[a, b]$. 