INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

Sample Examination, May 2001

MATH 3402

Functional Analysis

(UNIT COURSES)

Time: **TWO** (2) **hours** for working

Ten minutes for perusal before examination begins

Candidates should answer FOUR (**4**) questions.

All questions carry the same number of marks.

1.(a) For $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $p > 1$, define

$$||\underline{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} .$$

Prove that for $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^n$,

$$||\underline{x} + \underline{y}||_p \leq ||\underline{x}||_p + ||\underline{y}||_p$$

given that

$$\sum_{i=1}^{n} |x_i y_i| \leq ||\underline{x}||_p \, ||\underline{y}||_q \text{ where } \frac{1}{p} + \frac{1}{q} = 1 .$$

(b) The norm $||\underline{x}||_\infty$ is defined on $\mathbb{R}^n$ by

$$||\underline{x}||_\infty = \max_{1 \leq i \leq n} |x_i| .$$

Show that

$$||\underline{x}||_\infty \leq ||\underline{x}||_p \leq c||\underline{x}||_\infty$$

for some constant c which depends on both p and n.

2. Let $(X, \mathcal{T}_Y)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

(a) What is meant by the statement that the function $f : X \to Y$ is *continuous* ?

(b) Define a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.

(c) Show that, with the usual topology on $\mathbb{R}$, $(0, 1)$ is **not** homeomorphic to $[0, 1]$.

3.(a) Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$.
What is meant be the statements:
(i) '$\mathcal{C}$ is an open cover for A' ;
(ii) 'A is compact' .

(b) Prove that if A is compact, then any sequence $\{x_n\}$ in A has a limit point in A.

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4. (a) Let (X, d) be a metric space.
Define a **contraction mapping** on (X, d).

(b) Consider the mapping
$$f\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} .$$
Show that this is a contraction mapping on $\ell^2(2)$ but not on $\ell^1(2)$.
What is the fixed point of this mapping?

5. (a) State Riesz's lemma.

(b) The set $S = \{x \in X : ||x|| = 1\}$ in the normed linear space X is compact. Prove that X is finite dimensional.