MATH 3402

FIRST SEMESTER EXAMINATION, JUNE 2001

ASSORTED SOLUTIONS

1. Let (X, d) is a metric space. Define a new function $d' : X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \ .$$

Show that d' is a metric on X, and that (X, d') is bounded.

For $r > 0$, define

$$B_r(c) = \{x \in X;\ d(x, c) < r\}$$
$$B'_r(c) = \{x \in X;\ d'(x, c) < r\}$$

Show that for any $R > 0$ there is an $S > 0$ such that

$$B'_S(c) \subset B_R(c)$$

and conversely, for any $s > 0$ there is an $r > 0$ such that

$$B_r(c) \subset B'_s(c) \ .$$

Ans.

If $d(x, c) < r$, then

$$d'(x, c) = \frac{d(x, c)}{1 + d(x, c)} \leq d(x, c) < r$$

also. Therefore $B_r(c) \subset B'_r(c)$.

Given $R > 0$, let $S = R/(1 + R)$. Then, if $d'(x, c) < S$,

$$\frac{d(x, c)}{1 + d(x, c)} < \frac{R}{1 + R}$$
$$(1 + R)d(x, c) < R(1 + d(x, c))$$
$$d(x, c) < R$$

and $B'_S(c) \subset B_R(c)$.

2. Let Y_i, $i \in \mathbb{N}$, be subsets of a topological space $(X, \mathcal{T})$.
Show that

$$Z = \bigcap_{n=1}^{\infty} Cl\left(\bigcup_{i=n}^{\infty} Y_i\right)$$

is a closed subset of X.

Ans.

Since the closure of a set is closed, the sets

$$Cl\left(\bigcup_{i=n}^{\infty} Y_i\right)$$

are closed subsets of X.

Since the intersection of any number of closed sets is closed, Z is also closed.

If $\{x_n\}$ is a convergent sequence in X, converging to ζ, such that $x_n \in Y_n$ for all $n \in \mathbb{N}$, show that $\zeta \in Z$.

Ans.
Let $U_n = \cup_{i=n}^{\infty} Y_i$.
Since $x_n \in Y_n$ for all n, U_n contains the subsequence

$$\{x_n, x_{n+1}, x_{n+2}, \dots\}$$

and hence $Cl(U_n)$ contains the limit ζ for each n.
Therefore $\zeta \in Z$.

3.(a) Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$.
What is meant by the statements:
(i) '$\mathcal{C}$ is an open cover for A' ;
(ii) 'A is compact' .

(b) Let C be the topological space consisting of the rational numbers together with the discrete topology.

Which subsets of C (if any) are compact?

Ans.
In any topological space, a finite set $A = \{x_i\}_{i=1}^{n}$ is compact, since given any open cover $\{U_\alpha\}$, we can choose n sets U_i such that $x_i \in U_i$ which cover A.
In any set with the discrete topology, singleton sets are open.
Therefore for any infinite set $A = \{x_\alpha\}$, the collection

$$\mathcal{C} = \{\{x_\alpha\}\}$$

is an open cover for A such that no proper subset of $\mathcal{C}$ covers A. Hence no infinite set is compact.

Therefore the compact subsets of C are those containing a finite number of elements.

Let C^* be the one point compactification of C.
Describe the topology of C^*, and verify that C^* is compact.

Ans. Let $C^* = \{y\} \cup C$. The topology of C^* consists of
(a) The topology of C – all subsets of C ; and
(b) all sets of the form $\{y\} \cup M$, $M \subset C$, where $C \setminus M$ is compact in C; that is finite.

If $\mathcal{C} = \{U_\alpha\}$ covers C, then at least one set contains y.
That is, there is a set $U_0 = \{y\} \cup M_0$ such that $C \setminus M_0$ is finite; $C \setminus M_0 = \{x_1, x_2, \dots x_n\}$.
For each x_i choose $U_i \in \mathcal{C}$ such that $x_i \in U_i$.
Then $\{U_0, U_1, \dots U_n\}$ covers C^*, and C^* is compact.