MATH 3402

Continuity and Uniform Continuity

Cover.

A collection $\mathcal{G} = \{G_\alpha \subset X\}$ covers a set S in X if $S \subset \cup_\alpha G_\alpha$.

If each of the sets G_α is open in X, we say that this is an open cover for S.

Compact Set.

A set $S \subset X$ is compact if and only if for every collection $\{G_\alpha\}$ of open sets in X which cover S, there is a finite subcollection $\{G_{\alpha_i}\}$, $i = 1 \ldots n$, such that

$$S \subset \cup_{i=1}^n G_{\alpha_i}.$$

(Every open cover has a finite subcover.)

A compact set is bounded.

Let S be a compact set in (X, d).

Choose some $a \in X$.

The sets $G_n = \{x \in X : d(x, a) < n; n \in \mathbb{N}\}$ are open in (X, d), and cover X. Therefore they cover S.

Therefore there is a finite set $\{G_{n_1}, G_{n_2}, \ldots, G_{n_k}\}$ which covers S.

Therefore $d(x, a) < \max(n_i)$ for every $x \in S$, and S is bounded.

A compact set is closed.

Let S be a compact set in (X, d), and let $a \in \setminus S$.

The sets $G_n = \{x \in X : d(x, a) > \frac{1}{n}; n \in \mathbb{N}\}$ are open in (X, d) and cover $X \setminus \{a\}$. Therefore they cover S.

Therefore there is a finite set $\{G_{n_1}, G_{n_2}, \ldots, G_{n_k}\}$ which covers S.

If $N = \max n_i$, $d(x, a) > \frac{1}{N}$ for all $x \in S$, and a is not an accumulation point of S.

Therefore S contains all its accumulation points, and is closed.

The converse of these results does not hold in general.

If d is the discrete metric and X is an infinite set, then any infinite subset $S \subset X$ is closed and bounded, but is not compact.

The collection $\{G_\alpha = \{\alpha\} : \alpha \in S\}$ is an open cover for S but no finite subcollection covers S.

However, the converse does hold in $(\mathbb{R}, |.|)$.

The Heine-Borel Theorem. A closed and bounded set in $\mathbb{R}$ is compact.

Theorem. *An infinite subset of a compact set has an accumulation point in the set.*

Proof.

Suppose that S is a compact set in (X, d), the set $T \subset S$ has no accumulation point in S.

Then for every $x \in S$, there is a neighbourhood $\mathcal{N}(x, \epsilon_x)$ for some $\epsilon_x > 0$ which contains at most one point of T (when $x \in T$.)

The collection $\{\mathcal{N}(x, \epsilon_x)\}$ is an open cover for S, therefore there is a finite subcover for S.

Since this sub-cover also covers T, and each set in the subcover contains at most one point of T, the set T is finite.

It follows that a Cauchy sequence in a compact set converges in the set.

Theorem. *If S is a compact set, and $f : (X, d_X) \to (Y, d_Y)$ is continuous on S, then f is uniformly continuous on S.*

Proof.
Given any $\epsilon > 0$, for every x in S there is a $\delta_x > 0$ such that

$$d_X(y, x) < \delta_x \Longrightarrow d_Y(f(y), f(x)) < \epsilon/2.$$

For each $x \in S$, define the set G_x by

$$G_x = \{y \in S; d_X(y, x) < \delta_x/2\}.$$

The collection $\{G_x\}$ is an open cover for S.
Therefore, there is a finite set $\{x_1, \dots, x_n\}$ such that $\{G_{x_i}\}$ is an open cover for S.
Let δ =min$(\delta_{x_i}/2)$.
Any $x \in S$ is in one of the $\mathcal{G}_{x_i}$; ie $d_X(x, x_i) < \delta_{x_i}/2(< \delta_{x_i})$.

$$d_X(y, x_i) \leq d_X(y, x) + d_X(x, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}.$$

Therefore

$$d_Y(f(x), f(x_i)) < \epsilon/2 \text{ and } d_Y(f(y), f(x_i)) < \epsilon/2$$

so that $d_Y(f(y), f(x)) < \epsilon$.
i.e. $y, x \in S$ and $d_X(y, x) < \delta \Longrightarrow d_Y(f(y), f(x)) < \epsilon$.

Consequently, if $\{x_n\}$ is a Cauchy sequence in a compact set S, and f is continuous on S then $\{f(x_n)\}$ is a Cauchy sequence in $f(S)$.

Proof.
Since f is uniformly continuous from S to $f(S)$, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$d_Y(f(y), f(x)) < \epsilon \;\forall\; x, y \in S \;;\; d_X(y, x) < \delta \;.$$

Given this δ, we can find $N \in \mathbb{N}$ such that

$$d_X(x_n, x_m) < \delta \;\forall\; n, m > N \;.$$

But then

$$d_Y(f(x_n), f(x_m)) < \epsilon \;\forall\; n, m > N \;.$$

Theorem. *If $S \subset X$ is compact, and $f : X \to Y$ is continuous on S then $f(S)$ is compact.*

Proof. Let $\{G_\alpha\}$ be any open cover for $f(S)$. Then $\{f^{-1}(G_\alpha)\}$ is an open cover for S.

But S is compact, therefore there is a finite subcover $\{f^{-1}(G_{\alpha_i})\}$ for S, and $\{G_{\alpha_i}\}$ is now a finite subcover for $f(S)$.

Combining these results, we see that if $\{x_n\}$ is Cauchy in S, and f is continuous on S, $\{f(x_n)\}$ converges in $f(S)$.

Since a compact set is closed and bounded, we also have as a consequence the following.

Corollary ; The extreme value theorem. *If S is compact, and $f; S \to \mathbb{R}$ is continuous on S, then there are $x_1, x_2 \in S$ such that*

$$f(x_1) \leq f(x) \leq f(x_2) \text{ for all } x \in S.$$