## Topologocal Spaces

We start with some set $X$ (the space) and consider the set $2^X$ of all subsets of $X$.

From $2^X$ we select a collection $\mathcal{T}$ if subsets of $X$ with the following properties:

$$\begin{aligned}
&\phi \in \mathcal{T} \\
&X \in \mathcal{T} \\
&\text{If } \ T_1 \& T_2 \in \mathcal{T} \ , T_1 \cap T_2 \in \mathcal{T} \\
&\text{For any collection } \ T_\alpha \in \mathcal{T} \ , \ \bigcup_\alpha T_\alpha \in \mathcal{T}
\end{aligned}$$

These rules obviously mimic the properties of open sets in a metric space. The set $\mathcal{T}$ is now taken to be the complete set of "open sets" in $X$. We call $\mathcal{T}$ a topology on $X$, and the joint structure $(X, \mathcal{T})$ is called a topological space.

If we have a metric on $X$, then the collection of open sets generated by that metric is a topology on $X$. However, we do not need to have a metric to generate a topology.

Consider for example the set $X$ consisting of all students in MATH 3402.

If we select $A_1$ as the set of all students whose surnames begin with the letters A to M, and $A_2$ as the set of allstudents whose surnames begin with the letters N to Z, then

$$\mathcal{T}_1 = \{\phi, A_1, A_2, X\}$$

is a topology on $X$.

Similarly, if we select $G_1$ as the set of all boys in the class, and $G_2$ as the set of all girls,

$$\mathcal{T}_2 = \{\phi, G_1, G_2, X\}$$

is another topology on $X$.

We can combine these two topologies by considering

$$\begin{aligned}
S_{11} &= A_1 \cap G_1 \\
S_{12} &= A_1 \cap G_2 \\
S_{21} &= A_2 \cap G_1 \\
S_{22} &= A_2 \cap G_2
\end{aligned}$$

and forming the collection of all possible unions of these sets:

$$\begin{gathered}
\mathcal{T}_3 = \{\phi, S_{11}, S_{12}, S_{21}, S_{22}, \\
A_1, A_2, G_1, G_2, S_{11} \cup S_{22}, S_{12} \cup S_{21} \\
\backslash S_{11}, \backslash S_{12}, \backslash S_{21}, \backslash S_{22}, X\}
\end{gathered}$$

which is another topology on $X$.

The topology $\mathcal{T}_3$ contains every set that is in $\mathcal{T}_1$.
We say that $\mathcal{T}_3$ is **finer** that $\mathcal{T}_1$, and that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_3$.
Similarly, $\mathcal{T}_3$ is finer that $\mathcal{T}_2$, and $\mathcal{T}_2$ is coarser than $\mathcal{T}_3$.
However, $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable in this way.

The topology $\mathcal{T} = 2^X$, which is generated by the discrete metric, is the finest of all. This topology is called the **discrete topology**.

At the other extreme, the topology consisting of $\phi$ and $X$, which is called the **indiscrete topology** is the coarsest of all.

Between these extremes, the various topologies on $X$ generate a lattice.

**Continuity.**
Let $f$ be a function from the topological space $(X, \mathcal{T}_X)$ into the topological space $(Y, \mathcal{T}_Y)$.
We say that $f$ is continuous if for every $T \in \mathcal{T}_Y$, $f^{-1}(T) \in \mathcal{T}_x$.

This definition mimics the result for metric spaces.

For example consider the function which assigns a mark to each student in this class.

The set $Y$ of possible marks is traditionally classified into

$$\begin{aligned} C_1 &: [0..19] \\ C_2 &: [20..44] \\ C_3 &: [45..49] \\ C_4 &: [50..64] \\ C_5 &: [65..74] \\ C_6 &: [75..84] \\ C_7 &: [85..100] \end{aligned}$$

and we obtain a topology for $Y$ by collecting all possible unions of these sets.

$$\mathcal{T}_Y = \{\phi, C_i, C_i \cup C_j, C_i \cup C_j \cup C_k, \ldots, Y\}$$

The mapping $f : (X, \mathcal{T}_3) \to (Y, \mathcal{T}_Y)$ will be continuous in the unlikely event that the inverse image of each element of $\mathcal{T}_Y$ is in $\mathcal{T}_3$.

Since

$$f^{-1}(C_i \cup C_j) = f^{-1}(C_i) \cup f^{-1}(C_j)$$

the function will be continuous if $f^{-1}(C_i) \in \mathcal{T}_3$ for $i = 1, .., 7$.

**Bases.**
The set $\{C_1, C_2, C_3, C_4, C_5, C_6, C_7\}$ is a very simple example of a **basis** for a topological space.

A basis for a topological space $(X, \mathcal{T})$ is a subset $\{B_\alpha\}$ of $\mathcal{T}$ with the property that every element of $\mathcal{T}$ can be expressed as the union of elements of the basis.
In this definition we admit the possibility of a null union.

We also speak of a basis for $X$.

A collection $\mathcal{B}$ of subsets of $X$ is a basis for $X$ if
1) $X$ is a union of sets from $\mathcal{B}$.
2) For $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is a union of sets from $\mathcal{B}$.

Given a basis $\mathcal{B}$ for $X$, the set

$$\mathcal{T} = \{U \in 2^X : U \text{ is a union of sets from } \mathcal{B}\}$$

is a topology for $X$, and $\mathcal{B}$ is a basis for $\mathcal{T}$.

As we have seen in the particular case studied above,to prove that a function $f$ from $(X, \mathcal{T}_1)$ to $(Y, \mathcal{T}_2)$ is continuous, it is sufficient to show that $f^{-1}(B_i)$ is open in $(X, \mathcal{T}_1)$ for every $B_i$ in some basis for $\mathcal{T}_2$.

In general bases contain enormous numbers of sets. However, this number is "less" than the number of sets in the topolgy.

For example, we can form a basis for the topology on a metric space $(X, d)$ by considering the set $\mathcal{B}$ of all $\epsilon$ neighbourhoods $\mathcal{N}(a, \epsilon)$ where $\epsilon le K$ for some positive $K$.

When $d$ is the discrete metric and $K < 1$, this generates the basis of singletons.

For $\mathbb{R}$ this generates a basis containing $c^2 = c(< 2^c)$ elements.

We can economise by using the density of $\mathbb{Q}$ in $\mathbb{R}$, and restrict the basis to $\{\mathcal{N}(r_1, r_2)\}$ where $r_1$ and $r_2$ are rational $(0 < r_2 < K)$, which is a countable basis.

**Sub-bases.**

A sub-basis for a topology $\mathcal{T}$ is a subcollection $\mathcal{S} \subset \mathcal{T}$ such that any set in $\mathcal{T}$ is a union of finite intersections of sets from $\mathcal{S}$.

In particular a basis can be represented as the intersection of sets from the sub-basis.

For example, $\{A_1, A_2, G_1, G_2\}$ are a sub-basis for $\mathcal{T}_3$, although in this case there is no advantage in considering the sub-basis instead of the basis.

We can also define a sub-basis for the set $X$ as **any** collection $\mathcal{S}$ of subsets of $X$. The collection of all finite intersections of sets from $\mathcal{S}$ together with $X$ (which could be considered as a "null intersection") forms a basis for $X$.

The topology generated by this basis is called the topology generated by $\mathcal{S}$, and is the weakest topology containing $\mathcal{S}$.

Consider the function $f$ from $X$ to $(Y, \mathcal{T}_Y)$.

The topology generated by $\{f^{-1}(S)\,;\, S \in \mathcal{T}_Y\}$ is the weakest topology on $X$ relative to which $f$ is continuous.