Subspaces.

If we have a topological space $(X, \mathcal{T})$, then for any non-empty subset $A \subset X$ we obtain a topology for A by taking

$$\mathcal{T}_A = \{U \cap A \,;\, U \in \mathcal{T}\} \ .$$

Unless otherwise specified, we assume that a subset of a topological space inherits this topology.

This induced topology preserves continuity.

Denote by i the inclusion map from A to X; that is $i(x) = x$ for $x \in A$.
This map is continuous.
If $U \in \mathcal{T}$, then $i^{-1}(U) = U \cap A \in \mathcal{T}_A$.

If f is a continuous function from $(X, \mathcal{T})$ to $(Y, \mathcal{T}_Y)$, then $f \circ i$ is continuous from $(A, \mathcal{T}_A)$ to $(Y, \mathcal{T}_Y)$.
If $U \in \mathcal{T}_Y$, then $f^{-1}(U) \in \mathcal{T}$.
Therefore $f^{-1}(U) \cap A \in \mathcal{T}_A$.
But $(f \circ i)^{-1}(U) = f^{-1}(U) \cap A$, so that $f \circ i$ is continuous as required.

The function g from $(Y, \mathcal{T}_Y)$ to $(A, \mathcal{T}_A)$ is continuous if and only if $i \circ g$ is continuous from $(Y, \mathcal{T}_Y)$ to $(X, \mathcal{T})$.

Suppose that g is continuous from Y to A. Then $i \circ g$ is a continuous function of a continuous function and therefore continuous.

Suppose that $i \circ g$ is continuous from Y to X.
Consider $V \in \mathcal{T}_A$. $V = U \cap A$ for some $U \in \mathcal{T}$.
Therefore $V = i^{-1}(U)$, and

$$g^{-1}(V) = g^{-1}(i^{-1}(U)) = (i \circ g)^{-1}(U) \in \mathcal{T}_Y \ .$$

Conversely, suppose $\mathcal{T}'$ is a topology on A with these properties.
In particular, take $(Y, \mathcal{T}_Y) = (A, \mathcal{T}')$ and $g = I$, the identity function.
Since I is trivially continuous, $i \circ I$ is continuous from $(A, \mathcal{T}')$ to $(X, \mathcal{T})$.
Therefore, for any $U \in \mathcal{T}$, $i^{-1}(U) = U \cap A \in \mathcal{T}'$.
Hence $\mathcal{T}_A \subset \mathcal{T}'$.
On the other hand, if we take $(Y, \mathcal{T}_Y) = (A, \mathcal{T}_A)$, and $g = I$, then since $i \circ I = i$ from $(A, \mathcal{T}_A)$ to $(X, \mathcal{T})$ is continuous, I is continuous from $(A, \mathcal{T}_A)$ to $(A, \mathcal{T}')$.
Therefore if $V \in \mathcal{T}'$, $I^{-1}(V) = V \in \mathcal{T}_A$, and $\mathcal{T}' \subset \mathcal{T}_A$.
Hence $\mathcal{T}' = \mathcal{T}_A$.

Product spaces.

Given two sets X_1 and X_2, the product $X_1 \times X_2$ is defined as the set

$$\{(x_1, x_2)\,;\ x_1 \in X_1\,,\ x_2 \in X_2\}\ .$$

For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Given a product, we recover the co-ordinates by means of the projections;

$$\begin{array}{ll} p_1 : X_1 \times X_2 \to X_1 & p_1(x_1, x_2) = x_1 \\ p_2 : X_1 \times X_2 \to X_2 & p_2(x_1, x_2) = x_2 \end{array}$$

and if f is a function from Y to $X_1 \times X_2$, then $p_1 \circ f$ from Y to X_1 and $p_2 \circ f$ from Y to X_2 are the co-ordinate maps.

Conversely, two functions g from Y to X_1 and h from Y to X_2 together determine the function f from Y to $X_1 \times X_2$

$$f(x) = (g(x), h(x))\ ,$$

and $p_1 \circ f = g$, $p_2 \circ f = h$.

If we now have topologies $\mathcal{T}_1$ on X_1, and $\mathcal{T}_2$ on X_2, what topology $\mathcal{T}$ do we need to impose on $X_1 \times X_2$ to ensure that f is continuous from $(Y, \mathcal{T}_Y)$ to $(X_1 \times X_2, \mathcal{T})$ if and only if $p_1 \circ f$ and $p_2 \circ f$ are continuous from $(Y, \mathcal{T}_Y)$ to $(X_i, \mathcal{T}_i)$.

If we take $(Y, \mathcal{T}_Y) = (X_1 \times X_2, \mathcal{T})$ and $f = I$, then we need $p_i \circ I = p_i$ continuous from $(X_1 \times X_2, \mathcal{T})$ to $(X_i, \mathcal{T}_i)$.

Therefore for every $U_i \in \mathcal{T}_i$, $p_i^{-1}(U_i) \in \mathcal{T}$.

Therefore, $\mathcal{T}$ contains $U_1 \times X_2$ for every $U_1 \in X_1$, and $X_1 \times U_2$ for every $U_2 \in X_2$.

Since $\mathcal{T}$ is a topology,

$$(U_1 \times X_2) \cap (X_1 \times U_2) = U_1 \times U_2 \in \mathcal{T}\ ,$$

and hence $\mathcal{T}$ also contains all unions of sets of this form.

If we denote by $\mathcal{T}'$ the set of all unions of sets of the form $U_1 \times U_2$, where $U_i \in \mathcal{T}_i$, then $\mathcal{T}'$ is a topology on $X_1 \times X_2$.

We prove this by showing that

$$\mathcal{B} = \{U_1 \times U_2 : U_i \in \mathcal{T}_i\}$$

is a basis for $\mathcal{T}'$.

a) $X_i \in \mathcal{T}_i$, so that $X_1 \times X_2 \in \mathcal{B}$.

b) $(U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2)$

If $U_i, V_i \in \mathcal{T}_i$, $U_i \cap V_i \in \mathcal{T}_i$, so that the intersection of any two elements in $\mathcal{B}$ is also in $\mathcal{B}$.

Hence $\mathcal{B}$ is a basis which generates $\mathcal{T}'$.

Now consider the identity map from $(X_1 \times X_2, \mathcal{T}')$ to $(X_1 \times X_2, \mathcal{T})$.

For $p_1 \circ I = p_1$ from $(X_1 \times X_2, \mathcal{T}')$ to $(X_1, \mathcal{T}_1)$,

if $U_1 \in \mathcal{T}_1)$, then $p_1^{-1}(U_1) = U_1 \times X_2 \in \mathcal{T}'$,

so that the mapping is continuous.

Similarly $p_2 \circ I$ is continuous from $(X - 1 \times X_2, \mathcal{T}')$ to $(X_2, \mathcal{T}_2)$.

Therefore if $\mathcal{T}$ ensures continuity, I is continuous from $(X_1 \times X_2, \mathcal{T}')$ to $(X_1 \times X_2, \mathcal{T})$, and $\mathcal{T} \subset \mathcal{T}'$.

Hence $\mathcal{T} = \mathcal{T}'$.

We call this topology on $X_1 \times X_2$ the **product topology**.

Note that W is open in $X_1 \times X_2$ with this topology, then W is a union of sets of the form $U_1 \times U_2$, $U_i \in \mathcal{T}_i$, so that if $(x, y) \in W$, $(x, y) \in U_1 \times U_2$ for some U_1 and U_2 open in $\mathcal{T}_1$ and $\mathcal{T}_2$.

We started by looking for a topology which would ensure that f from $(Y, \mathcal{T}_Y)$ would be continuous if and only if $p_i \circ f$ from $(Y, \mathcal{T}_Y)$ to $(X_i, \mathcal{T}_i)$ were continuous, and we have derived the product topology by looking at particular cases.
It remains to show that we have succeeded in general.

First we note that p_1 and p_2 are continuous on the appropriate spaces.
If $U \in \mathcal{T}_1$, then $p_1^{-1}(U) = U \times X_2 \in \mathcal{T}$, so that p_1 is continuous. Similarly for p_2.
It follows that if f is a continuous function from $(Y, \mathcal{T}_Y)$ to $(X_1 \times X_2, \mathcal{T})$, then $p_1 \circ f$ and $p_2 \circ f$ are continuous.
Conversely, if $p_1 \circ f$ and $p_1 \circ f$ are continuous, then for any $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$,

$$\begin{aligned} f^{-1}(U_1 \times U_2) &= f^{-1}((U_1 \times X_2) \cap (X_1 \times U_2)) \\ &= f^{-1}(U_1 \times X_2) \cap f^{-1}(X_2 \times U_2) \\ &= f^{-1}(p_1^{-1}(U_1)) \cap f^{-1}(p_2^{-1}(U_2)) \\ &= (p_1 \circ f)^{-1}(U_1) \cap (p_2 \circ f)^{-1}(U_2) \\ &\quad \in \mathcal{T}_Y \end{aligned}$$

since it is the intersection of two sets in $\mathcal{T}_Y$.
Since the $\{U_1 \times U_2\}$ are a basis for $\mathcal{T}$, it follows that f is continuous from Y to $X_1 \times X_2$.