Topological Spaces

Subspaces.

If we have a topological space \((X, T)\), then for any non-empty subset \(A \subset X\) we obtain a topology for \(A\) by taking

\[ T_A = \{ U \cap A ; U \in T \} . \]

Unless otherwise specified, we assume that a subset of a topological space inherits this topology.

This induced topology preserves continuity.

Denote by \(i\) the inclusion map from \(A\) to \(X\); that is \(i(x) = x\) for \(x \in A\).

This map is continuous.

If \(U \in T\), then \(i^{-1}(U) = U \cap A \in T_A\).

If \(f\) is a continuous function from \((X, T)\) to \((Y, T_Y)\), then \(f \circ i\) is continuous from \((A, T_A)\) to \((Y, T_Y)\).

The function \(g\) from \((Y, T_Y)\) to \((A, T_A)\) is continuous if and only if \(i \circ g\) is continuous from \((Y, T_Y)\) to \((X, T)\).

Suppose that \(g\) is continuous from \(Y\) to \(A\). Then \(i \circ g\) is a continuous function of a continuous function and therefore continuous.

Suppose that \(i \circ g\) is continuous from \(Y\) to \(X\).

Consider \(V \in T_A\). \(V = U \cap A\) for some \(U \in T\).

Therefore \(V = i^{-1}(U)\), and

\[ g^{-1}(V) = g^{-1}(i^{-1}(U)) = (i \circ g)^{-1}(U) \in T_Y . \]

Conversely, suppose \(T'\) is a topology on \(A\) with these properties.

In particular, take \((Y, T_Y) = (A, T')\) and \(g = I\), the identity function.

Since \(I\) is trivially continuous, \(i \circ I\) is continuous from \((A, T')\) to \((X, T)\).

Therefore, for any \(U \in T\), \(i^{-1}(U) = U \cap A \in T'\).

Hence \(T_A \subset T'\).

On the other hand, if we take \((Y, T_Y) = (A, T_A)\), and \(g = I\), then since \(i \circ I = i\) from \((A, T_A)\) to \((X, T)\) is continuous, \(I\) is continuous from \((A, T_A)\) to \((A, T')\).

Therefore if \(V \in T'\), \(I^{-1}(V) = V \in T_A\), and \(T' \subset T_A\).

Hence \(T' = T_A\).
Product spaces.

Given two sets $X_1$ and $X_2$, the product $X_1 \times X_2$ is defined as the set

$$\{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}.$$ 

For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Given a product, we recover the co-ordinates by means of the projections;

$$p_1 : X_1 \times X_2 \to X_1 \quad p_1(x_1, x_2) = x_1$$
$$p_2 : X_1 \times X_2 \to X_2 \quad p_2(x_1, x_2) = x_2$$

and if $f$ is a function from $Y$ to $X_1 \times X_2$, then $p_1 \circ f$ from $Y$ to $X_1$ and $p_2 \circ f$ from $Y$ to $X_2$ are the co-ordinate maps.

Conversely, two functions $g$ from $Y$ to $X_1$ and $h$ from $Y$ to $X_2$ together determine the function $f$ from $Y$ to $X_1 \times X_2$

$$f(x) = (g(x), h(x)),$$

and $p_1 \circ f = g$, $p_2 \circ f = h$.

If we now have topologies $T_1$ on $X_1$, and $T_2$ on $X_2$, what topology $T$ do we need to impose on $X_1 \times X_2$ to ensure that $f$ is continuous from $(Y, T_Y)$ to $(X_1 \times X_2, T)$ if and only if $p_1 \circ f$ and $p_2 \circ f$ are continuous from $(Y, T_Y)$ to $(X_1, T_1)$.

If we take $(Y, T_Y) = (X_1 \times X_2, T)$ and $f = I$, then we need $p_1 \circ I = p_1$ continuous from $(X_1 \times X_2, T)$ to $(X_1, T_1)$.

Therefore for every $U_i \in T_i$, $p_i^{-1}(U_i) \in T$.

Therefore, $T$ contains $U_1 \times X_2$ for every $U_1 \in X_1$, and $X_1 \times U_2$ for every $U_2 \in X_2$.

Since $T$ is a topology,

$$(U_1 \times X_2) \cap (X_1 \times U_2) = U_1 \times U_2 \in T,$$ 

and hence $T$ also contains all unions of sets of this form.

If we denote by $T'$ the set of all unions of sets of the form $U_1 \times U_2$, where $U_i \in T_i$, then $T'$ is a topology on $X_1 \times X_2$.

We prove this by showing that

$$B = \{U_1 \times U_2 : U_i \in T_i\}$$ 

is a basis for $T'$.

a) $X_1 \in T_i$, so that $X_1 \times X_2 \in B$.

b) $(U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2)$

If $U_i, V_i \in T_i, U_i \cap V_i \in T_i$, so that the intersection of any two elements in $B$ is also in $B$.

Hence $B$ is a basis which generates $T'$.

Now consider the identity map from $(X_1 \times X_2, T')$ to $(X_1 \times X_2, T)$.

For $p_1 \circ I = p_1$ from $(X_1 \times X_2, T')$ to $(X_1, T_1)$, if $U_1 \in T_1$, then $p_1^{-1}(U_1) = U_1 \times X_2 \in T'$,

so that the mapping is continuous.

Similarly $p_2 \circ I$ is continuous from $(X_1 \times X_2, T')$ to $(X_2, T_2)$.

Therefore if $T$ ensures continuity, $I$ is continuous from $(X_1 \times X_2, T')$ to $(X_1 \times X_2, T)$, and $T \subseteq T'$.

Hence $T = T'$.
We call this topology on $X_1 \times X_2$ the **product topology**.

Note that $W$ is open in $X_1 \times X_2$ with this topology, then $W$ is a union of sets
of the form $U_1 \times U_2$, $U_i \in T_i$, so that if $(x, y) \in W$, $(x, y) \in U_1 \times U_2$ for some $U_1$ and $U_2$ open in $T_1$ and $T_2$.

We started by looking for a topology which would ensure that $f$ from $(Y, T_Y)$
would be continuous if and only if $p_i \circ f$ from $(Y, T_Y)$ to $(X_i, T_i)$ were continuous,
and we have derived the product topology by looking at particular cases.

It remains to show that we have succeeded in general.

First we note that $p_1$ and $p_2$ are continuous on the appropriate spaces.

If $U \in T_1$, then $p_1^{-1}(U) = U \times X_2 \in T$, so that $p_1$ is continuous. Similarly for $p_2$.

It follows that if $f$ is a continuous function from $(Y, T_Y)$ to $(X_1 \times X_2, T)$, then
$p_1 \circ f$ and $p_2 \circ f$ are continuous.

Conversely, if $p_1 \circ f$ and $p_2 \circ f$ are continuous, then for any $U_1 \in T_1$ and $U_2 \in T_2$,

$$f^{-1}(U_1 \times U_2) = f^{-1}((U_1 \times X_2) \cap (X_1 \times U_2))$$
$$= f^{-1}(U_1 \times X_2) \cap f^{-1}(X_1 \times U_2)$$
$$= f^{-1}(p_1^{-1}(U_1)) \cap f^{-1}(p_2^{-1}(U_2))$$
$$= (p_1 \circ f)^{-1}(U_1) \cap (p_2 \circ f)^{-1}(U_2)$$
$$\in T_Y$$

since it is the intersection of two sets in $T_Y$.

Since the $\{U_1 \times U_2\}$ are a basis for $T$, it follows that $f$ is continuous from $Y$ to $X_1 \times X_2$. 