Separation

Suppose that we have a sequence $\{x_n\}$ in a topological space $(X, \mathcal{T})$.

In order to consider the convergence of such a sequence, we consider the corresponding convergence result for a metric space in terms of open sets:

namely: A sequence converges to a limit $l \in X$ if every (open) neighbourhood containing l contains all but a finite number of points of the sequence.

Consequently, we define convergence in $(X, \mathcal{T})$ by:

A sequence converges to a limit $l \in X$ if every open set containing l contains all but a finite number of points of the sequence.

However, this definition can have some unexpected consequences.

If $\mathcal{T} = \{\phi, X\}$, then every sequence in $(X, \mathcal{T})$ converges, and every $l \in X$ is a limit of the sequence.

Similarly, if we have a sequence in MATH 3402 which converges to an element of S_{11} (say) then **every** element of S_{11} is a limit of the sequence.

If we want convergent sequences to have unique limits in X, then we need to impose further restrictions on the topology of X.

Basicly the problem arises because there are distinct points x_1 and x_2 such that every open set containing x_1 contains x_2. Hence any sequence converging to x_2 converges to x_1 also.

(In the examples above, this works both ways, but this is not necessary.

If we consider the set $\{a, b\}$ with the topology $\{\phi, \{a\}, X\}$, then every open set containing b contains a, but not vice-versa.)

This means that x_1 is a limit point of the set $\{x_2\}$, and this latter set is not closed.

The simplest separation axiom is therefore to require that every set consisting of a single point be closed.

A topological space satisfying this axiom is called a T_1-space.

An equivalent formulation is to require that if x_1 and x_2 are distinct points, then there is an open set S_1 containing x_1 but not x_2.

If we have the first form of the axiom, then if $\{x_2\}$ is closed then $S_1 = \backslash\{x_2\}$ is an open set containing x_1 but not x_2.

Conversely, if we take the second form, then given $y \in X$, for each $x \neq y$ we can find an open set S_x which contains x but not y.

But then $\backslash\{y\} = \cup_x S_x$ is open, and $\{y\}$ is closed.

Hausdorff Spaces.

A more stringent separation axiom requires that if $x_1 \neq x_2$, there exist open sets U_1 and U_2 such that $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \phi$. (We say that U_1 and U_2 are disjoint.)

Spaces with this property are called T_2-spaces or Hausdorff spaces.

Spaces for which the topology is derived from a metric are Hausdorff spaces.

Given $x_1 \neq x_2$, $d(x_1, x_2) = r > 0$.

Then we can choose $U_i = \{d(x, x_i) < \frac{r}{2}\}$.

Every Hausdorff space is obviously a T_1-space, but some T_1-spaces are not Hausdorff spaces.

Regular Spaces.

A topological space is called **regular** if for every point x and each closed set V not containing x, there exist disjoint open sets U_1 and U_2 such that $x \in U_1$ and $V \subset U_2$.

Equivalently, for each point x and each open set U containing x, there is an open set U_1 containing x such that $Cl(U_1) \subset U$.

A regular space need not be a T_1-space.

For example, the set of students in MATH 3402 together with the topology $\mathcal{T}_3$ generated by $\{S_{11}, S_{12}, S_{21}, S_{22}\}$ is regular but not T_1.

Therefore it is usual to consider regular T_1-spaces.

Such spaces are denoted T_3-spaces.

Every T_3-space is Hausdorff, but not the converse.

A similar problem arises with the next class of spaces.

Normal Spaces.

A topological space is **normal** if for every pair of disjoint closed sets V_1 and V_2 there exist disjoint open sets U_1 and U_2 with $V_i \subset U_i$.

The same example as above shows that a normal space need not be a T_1-space.

A normal T_1-space is denoted a T_4-space.

Every T_4-space is T_3, but again the converse need not apply.

We will consider mainly Hausdorff spaces.

In a Hausdorff space, a convergent sequence has a unique limit.

Suppose that l is a limit of a sequence in X, and that l_1 is any other point of X.

Then there are disjoint open sets U_1 containing l and U_2 containing l_1.

Since l is a limit, all but a finite number of terms in the sequence belong to U_1, so that at most a finite number belong to U_2.

Therefore l_1 is not a limit for the sequence.

Any subspace of a Hausdorff space is Hausdorff.

If $Y \subset X$, and $(X, \mathcal{T})$ is Hausdorff, then for any two distinct points x_1 and x_2 in Y, (and hence in X), there are disjoint open sets U_1 and U_2 in $\mathcal{T}$ containing x_1 and x_2 respectively.

But then $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$ are disjoint open sets in the relative topology which contain x_1 and x_2 respectively.

Compact sets.

A subset S of a topological space $(X, \mathcal{T})$ is called compact if every open covering of S includes a finite subfamily which covers S.

Since a subset consisting of a single point is trivially compact, we see that if the space is not T_1, a compact set need not be closed.

If S is compact as a subset of X then it is compact when we consider S as the whole space and use the relative topology.

In this case we refer to S as a compact space.

A set $S \subset X$ is compact if and only if in every family of relatively closed subsets of S, the intersection of all of which is ϕ, there is a finite subfamily with the same property.

If $\{V_\alpha\}$ is a family of relatively closed subsets with null intersection, then $\{U_\alpha = S \setminus V_\alpha\}$ is an open cover for S.

Therefore there is a finite subcover for S if and only if there is a finite subfamily of the $\{V_\alpha\}$ with null intersection.

A closed subset of a compact space is compact.

Let S be a compact space, and C a closed subset of S.

If $\mathcal{F}$ is any collection of relatively closed subsets of C whose intersection is ϕ, then for each $F \in \mathcal{F}$, F is closed in S.

Since S is compact, there is a finite subfamily of $\mathcal{F}$ whose intersection is null, and hence C is compact.

We now consider compactness as applied to Hausdorff spaces.

If C_0 and C_1 are disjoint compact sets in a Hausdorff space X, there exist disjoint open sets W_i containing C_i respectively.

Consider first the case in which C_0 consists of a single point x_0 which is not in C_1.

(Such a set is trivially compact)

Then for every point $y \in C_1$, $x_0 \neq y$, and there are disjoint open sets U_y and V_y containing x_0 and y respectively.

The sets V_y cover C_1, and hence there is a finite subfamily $\{V_1, \ldots, V_n\}$ which covers C_1.

Let $\{U_1, \ldots, U_n\}$ be the corresponding U_y.

Then $W_0 = \cap_{i=1}^n$ is an open set containing x_0, disjoint from each of the V_i, and therefore from $W_1 = \cup_{i=1}^n V_i$, which is an open set containing C_1.

Now consider the case in which C_0 is an arbitary compact set disjoint from C_1.

For each x in C_0, the above construction gives disjoint open sets W_{0x} containing x and W_{1x} containing C_1.

The collection $\{W_{0x}\}$ is an open cover for C_0.

Therefore there is a finite subcollection $\{W_{01}, \ldots, W_{0n}\}$ which covers C_0.

Each element of the corresponding subcollection $\{W_{1i}\}$ contains C_1, so that $W_1 = \cap_{i=1}^n W_{1i}$ is an open set containing C_1 and disjoint from $W_0 = \cup_{i=1}^n W_{0i}$ which is an open set containing C_0.

A compact subset of a Hausdorff space is closed.

Suppose C is compact, and $x \in Cl(C)$.

If x is not in C, then there are disjoint open sets W_0 containing x and W_1 containing C.

Therefore x is not a limit point of C either, which is a contradiction.

Therefore, $x \in C$ and $C = Cl(C)$.

Locally Compact Spaces

A topological space is **locally compact** if every point has a neighbourhood whose closure is compact.

Any topological space $(X, \mathcal{T}_X)$ can be embedded in another topological space $(Y, \mathcal{T}_Y)$ having just one mopre point than X in such a way that Y is compact and the relative topology of X as a subset of Y is the original topology of X.

This process is called the **one-point compactification** of X.

Let y be any point distinct from the points of X.

Let $\mathcal{W}$ be the class of open sets W in X such that $X \setminus W$ is compact.

Since ϕ is compact, $X \in \mathcal{W}$.

Let $Y = X \cup \{y\}$.

We choose for the topology $\mathcal{T}_Y$ of Y the topology $\mathcal{T}_X$ of X together with the sets $W \cup \{y\}$ for $W \in \mathcal{W}$.

Note that $Y = X \cup \{y\}$ belongs to $\mathcal{T}_Y$ as required.

Now consider any open covering of Y.

Since it covers y, it must contain at least one set of the form $W_0 \cup \{y\}$, where $X \setminus W_0$ is compact in X.

Furthermore, $X \setminus W_0$ is covered by the relatively open sets $V \cap X$ where V is in the given covering for Y.

Since $X \setminus W_0$ is compact, a finite subfamily of these sets cover $X \setminus W_0$, and these sets together with $W_0 \cup \{y\}$ cover Y.

Therefore Y is compact.

If X is a T_1-space, so is Y, but it can happen that Y is not Hausdorff even though X is.

However, if X is a locally compact Hausdorff space, Y will be a Hausdorff space.

Since X as a subset of Y inherits the original topology of X, any two points in X have the Hausdorff property. Therefore it suffices to consider $x \in X$ and the point y.

Since X is locally compact, there is an open set U containing x such that $Cl(U)$ is compact.

But then $(X \setminus Cl(U)) \cup \{y\}$ is an open set in Y containing y and disjoint from U. Therefore Y is Hausdorff.

Conversely, if X and Y are both Hausdorff, X is locally compact.