Countability and Density

The set $\mathbb{Q}$ of rational numbers is countable.

This means that we can arrange them in a sequence

$$\{r_1,\ r_2,\ r_3,\ \ldots\}$$

which contains every rational number.

We do this by first ordering the positive rational numbers.
We construct the doubly infinite array

$$\begin{array}{ccccc} 1/1 & 2/1 & 3/1 & 4/1 & \ldots \\ 1/2 & 2/2 & 3/2 & 4/2 & \ldots \\ 1/3 & 2/3 & 3/3 & 4/3 & \ldots \\ 1/4 & 2/4 & 3/4 & 4/4 & \ldots \end{array}$$

which certainly contains every positive rational number.

We now order these numbers by starting at the top left hand corner and going up and down the diagonals sequentially, omitting the terms which have already been counted.

This gives

$$1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ 3,\ 4,\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4},\ \ldots$$

We now order $\mathbb{Q}$ by starting with 0 and then alternating the above sequence with its negatives, giving

$$0,\ 1,\ -1,\ 2,\ -2,\ \frac{1}{2},\ -\frac{1}{2},\ \ldots$$

Between any two rational number r_1 and r_2 there are infinitely many other rational numbers.

Suppose, without loss of generality, that $r_1 < r_2$.
For every pair of positive integers (m, n), we have

$$\begin{aligned} \frac{mr_1 + nr_2}{m+n} &= \frac{(m+n)r_1 + n(r_2 - r_1)}{m+n} \\ &= r_1 + \frac{n}{m+n}(r_2 - r_1) > r_1 \\ \frac{mr_1 + nr_2}{m+n} &= \frac{m(r_1 - r_2) + (m+n)r_2}{m+n} \\ &= r_2 - \frac{m}{m+n}(r_2 - r_1) < r_2 \end{aligned}$$

The set

$$\left\{\frac{mr_1 + nr_2}{m+n}\right\}$$

is therefore an infinite set of rational numbers between r_1 and r_2.

Between any two rational numbers there are also infinitely many irrational numbers.

We show this in the same fashion as the previous result.

Starting with any positive irrational number x, (for example, π, e, $\sqrt{2}$), we can constuct the infinite set

$$\left\{ \frac{mr_1 + nxr_2}{m + nx} \right\}$$

of irrational numbers between r_1 and r_2.

Between any two real numbers there is a rational number.

Consider two real numbers, $(x > y)$, whose decimal expansions are

$$x = x_0 \,.x_1 \, x_2 \, x_3 \, \ldots$$
$$y = y_0 \,.y_1 \, y_2 \, y_3 \, \ldots$$

where we choose the non-terminating form

$$a9999999$$

in preference to the terminating form

$$(a + 1)000000$$

Since $x > y$, there will be some number $n \geq 0$ such that

$$y_i = x_i \;\forall\; i < n$$
$$y_n < x_n$$

The rational number

$$r = x_0 \,.x_1 \, \ldots \, x_n$$

obviously satisfies $y < r < x$ as required.

Since r is itself a real number, there is another rational number between y and r and yet another between r and x.

Hence there are infinitely many rational numbers between y and x.

The rational numbers are dense in $\mathbb{R}$.

This means that for every real number x and every $\epsilon > 0$ there is a rational number r such that $|x - r| < \epsilon$.

In fact, from what we have seen, there are infinitely many rational numbers between $x - \epsilon$ and $x + \epsilon$.

From this it might appear that rational and irrational numbers are equally common. However, this is not the case. There are immeasurably more real numbers than rational numbers.

The real numbers cannot be counted!

What we will show is that the real numbers between 0 and 1 cannot be counted.

We assume to the contrary that we can arrange the real numbers between 0 and 1 as a sequence

$$\begin{aligned} x_1 &= 0.\, a_{11}\, a_{12}\, a_{13}\, \ldots \\ x_2 &= 0.\, a_{21}\, a_{22}\, a_{23}\, \ldots \\ x_3 &= 0.\, a_{31}\, a_{32}\, a_{33}\, \ldots \end{aligned}$$

As before, we choose the non-terminating form of the decimal expansion where we have a choice.

Now consider the number

$$y = 0.\, y_1\, y_2\, y_3\, \ldots$$

where $y_i = 1$ if $a_{ii} \neq 1$ and $y_i = 5$ if $a_{ii} = 1$. (The actual choice of numbers here is arbitrary)

y is obviously a real number between 0 and 1, and by construction $y \neq x_i$ for any i.

Therefore our assumption that the real numbers between 0 and 1 could be counted is false.