Inner Product Spaces

An inner product on a complex linear space X is a function $\langle x|y\rangle$ from $X \times X \to \mathbb{C}$ such that

$$\langle x_1 + x_2|y\rangle = \langle x_1|y\rangle + \langle x_2|y\rangle \tag{1}$$

$$\langle y|x\rangle = \overline{\langle x|y\rangle} \tag{2}$$

$$\langle x|\alpha y\rangle = \alpha\langle x|y\rangle \tag{3}$$

$$\langle x|x\rangle > 0 \text{ for } x \neq 0 \tag{4}$$

Consequently,

$$\langle \alpha x|y\rangle = \bar{\alpha}\langle x|y\rangle \tag{5}$$

$$\langle x|y_1 + y_2\rangle = \langle x|y_1\rangle + \langle x|y_2\rangle \tag{6}$$

Where the underlying field is $\mathbb{R}$, the conjugations can be omitted.

If X is an inner-product space, we define a norm on X by

$$||x|| = \sqrt{\langle x|x\rangle}\ ,$$

and we have

$$|\langle x|y\rangle| \leq ||x||\,||y||\ .$$

The spaces ℓ^2 and $L^2(a, b)$ are inner-product spaces with the inner products

$$\langle x|y\rangle = \sum_{i=1}^{\infty} \bar{\xi}_i \eta_i$$

and

$$\langle x|y\rangle = \int_a^b \overline{x(t)}y(t)\,dt$$

respectively.

Orthogonality. We say that x and y are *orthogonal* if $\langle x|y\rangle = 0$. (In complex spaces the term *unitary* is also used.)

If x and y are orthogonal, we write

$$x \perp y$$

Note that if $x \perp y$ then $y \perp x$, and that $x \perp 0$ for all $x \in X$.

If x is orthogonal to every element of a set S, we say that x is orthogonal to the set, and write $x \perp S$.

Since the inner product is linear, it follows that if $x \perp y$ and $x \perp z$ then $x \perp (\alpha y + \beta z)$ for all α, β.

Hence if $x \perp S$, x is also orthogonal to the space spanned by S.

Finally, since the inner product is continuous it follows that if $x \perp y_n$ and $y_n \to y$ then $x \perp y$.

Combined with the previous result we see that if $x \perp S$, x is orthogonal to the closure of the space generated by S.

A set $S = \{x_i\}$ of vectors is an orthogonal set if $x_i \perp x_j = 0$ for $i \neq j$.
If, in addition, $||x_i|| = 1$ for each i, we say that the set is orthonormal.
An orthonormal set of vectors is linearly independent.

Let $x = \sum_{i=1}^{n} \gamma_i u_i$ be any linear combination of vectors from an orthomormal set. Then

$$\langle u_j | x \rangle = \sum_{i=1}^{n} \gamma_i \langle u_j | u_i \rangle = \gamma_j$$

Therefore $x = 0$ if and only if $\gamma_i = 0 \ \forall i$, and the vectors are linearly independent.

Consequently, if $\{u_i\}_{i=1}^{n}$ is a finite orthonormal set, then it is a basis for the space M which it spans, and if $x \in M$,

$$x = \sum_{i=1}^{n} \langle u_i | x \rangle u_i \ .$$

Bessel's inequality
Let S be an orthonormal set in X.
If $u_1, \dots u_n$ is any finite collection of distinct elements of S, and $x \in X$, then

$$\sum_{i=1}^{n} |\langle u_i | x \rangle|^2 \leq ||x||^2 \ .$$

Write $\xi_i = \langle u_i | x \rangle$ and $y = x - \sum_{i=1}^{n} \xi_i u_i$. Then

$$0 \leq \langle y | y \rangle = \langle x | x \rangle - \sum_{i=1}^{n} \xi_i \langle x | u_i \rangle - \sum_{i=1}^{n} \bar{\xi}_i \langle u_i | x \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\xi}_i \xi_j \langle u_i | u_j \rangle$$

Since $\langle u_i | u_j \rangle = \delta_{ij}$ the last sum collapses, and since $\langle x | u_i \rangle = \bar{\xi_i}$ we have

$$0 \leq ||x||^2 - \sum_{i=1}^{n} |\xi_i|^2 - \sum_{i=1}^{n} |\xi_i|^2 + \sum_{i=1}^{n} |\xi_i|^2$$

as required.

It follows that for any fixed $x \in X$, the number of elements $u \in S$ for which $\langle x | u \rangle \neq 0$ is countable.
For any positive integer n, the number of elements u for which $|\langle x | u \rangle| > \frac{1}{n}$ is at most $n^2 ||x||^2$.
Therefore the number of u for which $\langle x | u \rangle \neq 0$ is a countable union of finite sets, which is countable.

Finally, from Cauchy's (Hölder's) inequality

$$\sum_{i=1}^{n} |\langle x | u_i \rangle \langle u_i | y \rangle| \leq \left(\sum_{i=1}^{n} |\langle x | u_i \rangle|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |\langle y | u_i \rangle|^2 \right)^{1/2} \leq ||x|| \, ||y|| \ .$$

The previous results have not assumed that the orthonormal set is countable.

While in almost all cases of interest this will be the case, it is possible to construct uncountable orthonormal sets.

For example, we can construct such a set by considering any uncountable set Q and taking as our vector space the set X of all complex valued functions bounded on Q which vanish almost everywhere.

If Q_x is the set of points $q \in Q$ at which $x(q) \neq 0$, and Q_y is the set of points at which $y(q) \neq 0$, we can define

$$\langle x|y\rangle = \sum_{q \in Q_x \cap Q_y} \overline{x(q)} y(q) \ .$$

The set of funtions $u_p(q) = \delta_{pq}, p \in Q$ is now an uncountable orthonormal set in X.

If however X contains a countable dense subset Y, (X is separable) then any orthonormal set S in X is countable.

We start by remarking that if $x \neq u \in S$,then

$$\langle x-u|x-u\rangle = \langle x|x\rangle - \langle x|u\rangle - \langle u|x\rangle + \langle u|u\rangle = 2$$

so that $||x-u|| = \sqrt{2}$ for distinct members of an orthonormal set.

Now suppose that $\{y_n\}$ is a countable dense subset of X, and consider any two distinct elements x, u of S.

Since this subset Y is dense in X, given $\epsilon = \sqrt{2}/3$, we can find y_{n_1} and y_{n_2} such that

$$\begin{aligned} ||x - y_{n_1}|| &< \frac{\sqrt{2}}{3} \\ ||u - y_{n_2}|| &< \frac{\sqrt{2}}{3} \end{aligned}$$

But then

$$\begin{aligned} \sqrt{2} = ||x-u|| &\leq ||x - y_{n_1}|| + ||y_{n_1} - y_{n_2}|| + ||y_{n_2} - u|| \\ &< \frac{2\sqrt{2}}{3} + ||y_{n_1} - y_{n_2}|| \\ ||y_{n_1} - y_{n_2}|| &> \frac{\sqrt{2}}{3} \\ y_{n_1} &\neq y_{n_2} \end{aligned}$$

and there is a $1-1$ correspondence between the elements of S and a subset of Y. Therefore S is countable.

Suppose that the space X is complete, and that $\{u_i\}$ is a countably infinite orthonormal set in X.

Then a series of the form

$$\sum_{i=1}^{\infty} \xi_i u_i$$

is convergent if and only if

$$\sum_{i=1}^{\infty} |\xi|^2 < \infty \ .$$

If the sum of the series is x, then

$$\xi_i = \langle u_i | x \rangle \ .$$

If $s_n = \sum_{i=1}^{n} \xi_i u_1$, and $m > n$, then

$$||s_m - s_n||^2 = \left|\left| \sum_{i=n+1}^{m} \xi_i u_i \right|\right|^2 = \sum_{i=n+1}^{m} |\xi_i|^2$$

and the sequence $\{s_n\}$ is Cauchy in X (and therefore converges since X is complete) if and only if $\sum |\xi_i|^2$ converges.

Since $\xi_i = \langle u_i | s_n \rangle$ for $1 \leq i \leq n$, and s_n converges to x, we have

$$\langle u_i | x \rangle = \lim_{n \to \infty} \langle u_i | s_n \rangle = \xi_i \ .$$

Note that $\sum |\xi_i|^2$ converges absolutely, and therefore the convergence is unaffected by arbitrary reordering of the terms.

This means that the convergence of $\sum \xi_i u_i$ is also independent of the ordering of the u_i.

Suppose that S is an orthonormal set in the complete inner-product space X.

If $x \in X$, then there are at most a countable number of elements of S for which $\langle u | x \rangle \neq 0$.

Index these by $\{u_i\}$ and consider

$$\sum_i \langle u_i | x \rangle u_i \ .$$

Since

$$\sum_{i=1}^{n} |\langle u_i | x \rangle|^2 \leq ||x||^2 \text{ for each } n,$$

$$\sum_{i=1}^{\infty} |\langle u_i | x \rangle|^2 < \infty \ ,$$

and

$$\sum_i \langle u_i | x \rangle u_i$$

converges to some element x_S in the closed manifold generated by S.

By construction, $(x - x_S) \perp u_i$ for each element in the subset, and for other elements $u \in S$, $\langle u | x \rangle = 0$ and $\langle u | u_i \rangle = 0$ so that $\langle u | x_S \rangle = 0$ also.

Therefore $(x - x_S) \perp S$ and hence $(x - x_s) \perp M$.

Complete orthonormal sets.

An orthonormal set $S \subset X$ is complete in X if there is no orthonormal set in X of which S is a proper subset.

Provided the inner-product space X is complete and is not the zero space, we can construct a complete orthonormal set in X, and if S is an orthonormal set in X, we can construct this set so that S is a subset.

We proceed inductively:

Since X contains at least one non-zero vector x_1, it contains the normal vector $u_1 = x_1/||x_1||$. The set $S_1 = \{u_1\}$ is an orthonormal set in X.

Suppose that S is any orthonormal set in X. Let M be the closed linear manifold generated by S.

If $M = X$, $x = \sum \langle u_i|x\rangle u_i$ for each $x \in X$, and therefore $x \perp S$ if and only if $x = 0$. Therefore S is complete.

Otherwise, we can find some vector $x \in X \setminus M$, and $(x - x_S) \perp S$.

Setting $u = (x - x_S)/||x - x_S||$, $S \cup \{u\}$ is an orthonormal set containing S.

That this process terminates is a consequence of Zorn's Lemma.

If the space X is separable then there is a countable complete orthonormal set S in X.

Since X is separable, there is a countable set $\{x_n\}$ which is dense in X.

Let y_1 be the first nonzero element in this set, y_2, the first (nonzero) element which is not in the space generated by y_1, and in general y_{k+1} the first element which is not in the space generated by $\{y_1, \ldots y_k\}$.

The set $\{y_m\}$ generates the same linear manifold M as $\{x_n\}$, and since the set $\{x_n\}$ is dense in X, $M = X$.

Applying the Gram-Schmidt process to $\{y_m\}$ now generates an orthonormal set S which is countable, and since it generates X it is complete.

Parseval's Formula.

Let S be an orthonormal set in X.

If
$$||x||^2 = \sum_{u \in S} |\langle u|x\rangle|^2$$
for every $x \in X$, then S is complete.

If $x \perp S$, then $||x||^2 = 0$, $x = 0$, and hence S is complete.

Hilbert Spaces.

A Hilbert space is a complete, infinite dimensional inner-product space.

In most applications, the space is also separable so that there is a countable orthonormal basis for the space.

However, as we have seen it is possible to produce spaces with uncountable orthonormal sets.

The spaces $L^2(a,b)$ are separable; the usual orthonormal sets for these spaces are given in terms of the classical "orthogonal polynomials":

For $L^2(-1,1)$, the Legendre polynomials

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1)\right)^n$$

satisfy

$$\int_{-1}^{1} P_n(t)P_m(t)\,dt = 0 \;\forall\; n \neq m \;.$$

The normalised set $\{\sqrt{n+\frac{1}{2}}P_n(t)\}$ is an orthonormal basis.

For other finite intervals we can scale the variables appropriately.

For $L^2(0,\infty)$, the Laguerre polynomials

$$L_n(t) = e^t\frac{d^n}{dt^n}\left(t^n e^{-t}\right)$$

satisfy

$$\int_0^\infty e^{-t}L_n(t)L_m(t)\,dt = 0 \;\forall\; n \neq m \;.$$

The set $\{\phi_n(t)\}$ where

$$\phi_n(t) = \frac{1}{n!}e^{-t/2}L_n(t)$$

is a complete orthonormal set for this space.

Finally for $L^2(-\infty,\infty)$, we have the Hermite polynomials

$$H_n(t) = (-1)^n e^{t^2}\frac{d^n}{dt^n}e^{-t^2}$$

which satisfy

$$\int_{-\infty}^{\infty} e^{-t^2}H_n(t)H_m(t)\,dt = 0\forall\; n \neq m \;.$$

In this case the orthonomal set is $\{(2^n n!\sqrt{\pi})^{-1/2}e^{-t^2/2}H_n(t)\}$.