Sets

We begin by revising the basic properties of sets; particularly sets of numbers. A set is a collection of objects from some *universe* U; numbers, functions; defined by some rule.

We usually denote sets by

 $S = \{(x \in U;)x \text{ has some property }\}$

although occasionally we may merely list the members of the set.

The universe may not always be specified if it is obvious from the context.

Some basic universes are:

 $\mathbb N~$: the natural numbers

- $\mathbbm{Z}\ :$ the integers
- \mathbb{Q} : the rational numbers
- ${\mathbb R}\,$: the real numbers
- $\mathbb C~$: the complex numbers

The set containing no elements, the null set or empty set, is denoted by

For example:

$$\{x \in \mathbb{N}; x = 2y, y \in \mathbb{N}\}$$

$$\{(x_n \in \mathbb{Q};)x_0 = 1, x_{n+1} = 1 + 1/x_n, n \ge 0\}$$

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INTERVALS OF THE REAL LINE

One class of sets which occur frequently are the real intervals

$$\begin{array}{l} (a,b) \ : \{a < x < b\} \\ (a,b] \ : \{a < x \le b\} \\ [a,b) \ : \{a \le x < b\} \\ [a,b] \ : \{a \le x < b\} \\ (-\infty,b) \ : \{x < b\} \\ (-\infty,b) \ : \{x \le b\} \\ (a,\infty) \ : \{x > a\} \\ [a,\infty) \ : \{x \ge a\} \end{array}$$

Note that the round brackets (,) indicate that the corresponding endpoint is not in the set, while the square brackets [,] indicate that the endpoint is included.

UNION AND INTERSECTION

The union

 $A\cup B$

of two sets A and B is the set

$$\{x \in A \text{ or } x \in B\}$$

e.g. If A = [0, 2] and $B = (1, 3), A \cup B = [0, 3)$. The intersection $A \cap B$

of two sets A and B is the set

$$\{x \in A \text{ and } x \in B\}$$

e.g. If A = [0,2] and $B = (1,3), A \cap B = (1,2]$ If A = (0,1] and $B = (1,2), A \cap B = \emptyset$

If we have a collection of sets A_i indexed in some way, then

$$\bigcup_i A_i$$

is the set of elements in some A_i , and

$$\bigcap_i A_i$$

is the set of elements in every A_i .

Complements

The set

 $A \setminus B$

denotes the set of elements which are in A but not in B.

e.g. If A = [0, 2] and B = (1, 3),

$$A \setminus B = [0, 1]$$

and

$$B \setminus A = (2,3)$$

Where the set A is the universe, it is usually omitted from the descriptor , and we have

 $\setminus B$

for the set of elements not in B.

This set is called the *complement* of B.

e.g. If the universe is $\mathbb{R},\,\backslash\mathbb{Q}$ is the set of irrational numbers.

If the universe is [0, 1], \mathbb{Q} is the set of irrational numbers between 0 and 1.

DE MORGAN'S LAWS

The processes of union and intersection are related under complementation by the rules

$$\setminus \left(\bigcup_{i} A_{i}\right) = \bigcap_{i} \left(\setminus A_{i}\right)$$
$$\setminus \left(\bigcap_{i} A_{i}\right) = \bigcup_{i} \left(\setminus A_{i}\right)$$

which are known as De Morgan's laws.

Subsets

A is a subset of B if every element of A is also an element of B. We write

 $A\subset B$

Given a universe U, then every set A of elements of U is a subset of U, and \emptyset is a subset of every set A.

i.e.

$$\emptyset \subset A \subset U$$

FUNCTIONS

If f is a function from one universe U to another universe V, then for $A \subset U$,

f(A)

is the set

$$\{y \in V; y = f(x), x \in A\}$$

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e.g.

If f from \mathbb{R} to \mathbb{R} is given by $f(x) = x^2$,

$$f((-1,1)) = [0,1)$$

Similarly, for $B \subset V$,

 $f^{-1}(B)$

is the set

$$\{x \in U; f(x) = y \text{ for some } y \in B\}$$

e.g. For f as before,

if
$$B = (1,4]$$
, $f^{-1}(B) = [-2,-1) \cup (1,2]$

and

if
$$B = (-2, -1)$$
, $f^{-1}(B) = \emptyset$

CONVERGENCE, COMPLETENESS AND CAUCHY SEQUENCES

In order to introduce some of the important concepts which arise in this course, we consider the double sequence defined by

$$p_1 = 1$$

$$q_1 = 1$$

$$p_{n+1} = p_n + 2q_n$$

$$q_{n+1} = p_n + q_n$$

so that

$$p_{2} = 1 + 2 = 3$$

$$q_{2} = 1 + 1 = 2$$

$$p_{3} = 3 + 4 = 7$$

$$q_{3} = 2 + 3 = 5 \qquad etc$$

We can show that

$$p_n \ge 2^{n-1} ; q_n \ge 2^{n-1} \forall n \ge 1$$

$$p_1 = 1 = 2^0$$

 $q_1 = 1 = 2^0$

so that the proposition is true for n = 0.

Suppose that the proposition is true for $1 \le n \le k$. In particular, it is true for n = k. Then

$$p_{k+1} = p_k + 2q_k \ge 2^{k-1} + 2^k > 2^k$$
$$q_{k+1} = p_k + q_k \ge 2^{k-1} + 2^{k-1} = 2^k$$

so that it is also true for n = k + 1.

Therefore, by the principle of mathematical induction, the proposition is true for all integers $n \ge 1$.

Now consider

$$p_{n+1}^2 - 2q_{n+1}^2 = (p_n + 2q_n)^2 - 2(p_n + q_n)^2$$

= $p_n^2 + 4p_nq_n + 4q_n^2$
- $2p_n^2 - 4p_nq_n - 2q_n^2$
= $-p_n^2 + 2q_n^2 = (-1)(p_n^2 - 2q_n^2)$
= $(-1)^2(p_{n-1}^2 - 2q_{n-1}^2) \dots$
= $(-1)^n(p_1^2 - 2q_1^2) = (-1)^{n+1}$

Since

$$p_n^2 - 2q_n^2 = (-1)^n$$
$$\frac{p_n^2}{q_n^2} - 2 = \frac{(-1)^n}{q_n^2}$$
$$\left| \left(\frac{p_n}{q_n}\right)^2 - 2 \right| = \frac{1}{q_n^2}$$
$$\leq \frac{1}{2^{2n-2}}$$

Given any $\epsilon > 0$, we can determine N such that

$$\frac{1}{2^{2N-2}} < \epsilon$$

Specifically, we choose

$$2^{2N-2} > \frac{1}{\epsilon}$$
$$(2N-2)\log 2 > -\log \epsilon$$
$$N > 1 - \frac{1}{2}\frac{\log \epsilon}{\log 2}$$

e.g. If $\epsilon = 10^{-6}$, we can take N = 11.

For any n > N, $2^{2n-2} > 2^{2N-2}$, so that

$$\left| \left(\frac{p_n}{q_n} \right)^2 - 2 \right| \le \frac{1}{2^{2n-2}}$$
$$< \frac{1}{2^{2N-2}} < \epsilon$$

and the sequence

$$\left\{ \left(\frac{p_n}{q_n}\right)^2 \right\}$$

converges to 2.

What about the sequence

$$\left\{\frac{p_n}{q_n}\right\} ?$$

Is there some number l such that, given any $\epsilon>0,$ we can find an integer N such that

$$\left|\frac{p_n}{q_n} - l\right| < \epsilon$$

for all n > N ?

Note that, if there is such a number, we have (rather crudely)

$$\begin{aligned} \frac{p_n}{q_n} &< 2\\ l &< 2 \end{aligned}$$
$$\left| \left(\frac{p_n}{q_n} \right)^2 - l^2 \right| = \left| \frac{p_n}{q_n} - l \right| \left| \frac{p_n}{q_n} + l \right| \\ &< 4 \left| \frac{p_n}{q_n} - l \right| < 4\epsilon \ \forall \ n > N \end{aligned}$$

so that p_n^2/q_n^2 converges to l^2 , and therefore $l^2 = 2$.

The Greek option

If the only numbers which you recognise are rational numbers; i.e. the ratios of integers; then this sequence does not converge.

There is no rational number p/q such that

$$\frac{p^2}{q^2} = 2 \ .$$

Suppose that there were such a number.

We may assume that we have cancelled any common factors, so that p and q are relatively prime.

However, if

$$p^2 = 2q^2$$

it follows that p must be even, since the square of an odd number is odd. Therefore, p = 2r for some integer r.

Substituting we have

$$4r^2 = 2q^2$$
$$q^2 = 2r^2$$

The same argument now shows that q is also even, which contradicts our assumption that p and q are relatively prime.

Historically, this question first arose when the Pythagoreans were considering the diagonal of a unit square.

If this diagonal has length l, then by Pythagoras' Theorem, $l^2 = 2$.

The Greek response was to refer to this length as incommensurable.

They also realised that this was not an isolated phenomenon, and developed tools for manipulating these quantities in terms of rational numbers.

The alternative approach, adopted over the next 2000 years, was to assume that all line segments could be measured in terms of *real numbers*, which obeyed the same algebraic laws as the rational numbers.

This process of filling in the 'holes' in the rational numbers is referred to as *completion*.

Since convergence requires a limit, we need some mechanism for dealing with sequences for which the terms get closer and closer together, but for which it may not be clear that a limit exists.

Cauchy sequences

If the sequence $\{a_n\}$ converges to l, then given any $\epsilon>0$ we can find an integer N such that

$$|a_n - l| < \frac{\epsilon}{2} \forall n > N .$$

If n and m are integers greater than N,

$$|a_n - a_m| = |(a_n - l) - (a_m - l)|$$

$$\leq |a_n - l| + |a_m - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

A sequence with the property

"Given any $\epsilon > 0$ there is an integer N such that

$$|a_n - a_m| < \epsilon \ \forall \ m, n > N$$

is called a Cauchy sequence.

We have seen that a convergent sequence is necessarily a Cauchy sequence.

On the other hand, the sequence $\{p_n/q_n\}$ considered as a sequence of rational numbers is a Cauchy sequence which does not converge.

Consider

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{p_n + 2q_n}{p_n + q_n} - \frac{p_n}{q_n}$$
$$= \frac{p_n q_n + 2q_n^2 - p_n^2 - p_n q_n}{(p_n + q_n)q_n}$$
$$= \frac{2q_n^2 - p_n^2}{q_{n+1}q_n}$$
$$= \frac{(-1)^{n+1}}{q_{n+1}q_n}$$
$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \le \frac{1}{2^{2n-1}}$$

If m > n,

$$\left|\frac{p_m}{q_m} - \frac{p_n}{q_n}\right| \le \sum_{r=n}^{m-1} \left|\frac{p_{r+1}}{q_{r+1}} - \frac{p_r}{q_r}\right|$$
$$< \sum_{r=n}^{m-1} \frac{1}{2^{2r-1}}$$
$$< \sum_{r=n}^{\infty} \frac{1}{2^{2r-1}} = \frac{1}{2^{2n-1}} \frac{4}{3}$$

Given $\epsilon > 0$, we choose N so that

$$\frac{1}{3 \, 2^{2N-3}} < \epsilon$$

and then

$$\left|\frac{p_m}{q_m} - \frac{p_n}{q_n}\right| < \epsilon$$

for all m > n > N .

The property of a set that all Cauchy sequences converge is called *completeness*. As we shall see, the real numbers are complete but the rational numbers are not.

Real numbers

We have seen that a convergent sequence is a Cauchy sequence.

We have also seen that a Cauchy sequence in \mathbb{Q} need not converge in \mathbb{Q} .

However, we expect that this is not the case in \mathbb{R} .

In order to prove that Cauchy sequences in \mathbb{R} converge, we need first to consider the distinguishing property of the real numbers.

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The real numbers \mathbb{R} can be visualised as points on the number line. They are distinguished from the rational numbers \mathbb{Q} by the following property, which we take as a defining axiom:

Least upper bound axiom.

A non-empty set of real numbers which is bounded above has a least upper bound.

That is: If we have a non-empty set S of real numbers x such that, for some number $K, x \leq K$ for every $x \in S$, then **there is a real number** l such that

(i) $x \leq l$ for every $x \in S$;

(ii) If $x \leq k$ for every $x \in S$, then $l \leq k$.

This least upper bound is denoted by *l.u.b.* or sup (for supremum). The sup may or may not be a member of the set. For example, 1 is the sup of both (0,1) and [0,1].

This property distinguishes the real numbers from the rational numbers. For example, the set of numbers

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

is a set of rational numbers.

Furthermore

$$\left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^n \frac{n!}{(n-i)!i!} \left(\frac{1}{n}\right)^n$$

$$= 1 + n\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \dots$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right)$$

$$+ \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$< 3$$

so that these rational numbers are bounded above.

However this set does not have a **rational** least upper bound.

The least upper bound of this set, considered as real numbers, is e, which is not rational.

An immediate consequence of the least upper bound axiom is that a non-empty set of real numbers which is bounded below has a greatest lower bound (g.l.b.) or infimum (inf).

(Consider the set $\{-x; x \in S\}$.)

If the set contains only finitely many numbers, then the supremum is the largest number in the set and the infimum is the least.

The Archimedean Principle.

For any two positive real numbers a and b, there is a positive integer n such that na > b.

We prove this result by showing that its negation leads to a contradiction.

Suppose that no such integer exists.

Then the set $\{na; n \in \mathbb{N}\}$ is non-empty and bounded above by b.

Therefore this set has a l.u.b l.

Since *l* is the l.u.b., $l - \frac{1}{2}a$, which is less than *l*, is not an upper bound. Therefore, for some *N* we must have

$$l - \frac{1}{2}a < Na \le l \; .$$

But now

$$(N+1)a = Na + a > l + \frac{1}{2}a > l$$

which contradicts the condition that l is an upper bound for the set.

Some consequences of this result are:

If $\epsilon > 0$, there is an integer N such that $\epsilon > \frac{1}{N}$.

If $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$, then x = 0.

Nested intervals.

Consider an infinite set of closed intervals

$$I_n = \{a_n \le x \le b_n\}$$

in \mathbb{R} with the following two properties.

(i) $I_{n+1} \subset I_n$; i.e. $a_n \le a_{n+1} < b_{n+1} \le b_n$

(ii) $b_n - a_n \to 0$ as $n \to \infty$.

(We call these a set of nested intervals.)

Then there is precisely one real number l which lies in every interval.

(This result asserts both existence and uniqueness.)

Proof: Consider firstly the set $\{a_m\}$. This set is bounded above by b_n for any n. Therefore this set has a least upper bound l_1 which has the two properties

$$a_n \leq l_1 \leq b_n \quad \text{for all } n \; .$$

The set $\{b_n\}$ is therefore bounded below by l_1 , so that there is a greatest lower bound l_2 which has the properties

$$l_1 \leq l_2 \leq b_n$$
 for all n .

Combining these results we see that

$$0 \le l_2 - l_1 \le b_n - a_n \quad \text{for all } n,$$

and since $b_n - a_n \to 0$ as $n \to \infty$, we must have $l_2 = l_1$.

This common limit is the required number.

Since $a_n \leq l_1 \leq b_n$ for each *n*, it lies in every interval.

To show uniqueness, consider any other number m.

If $m < l_1$, then m is not an upper bound for $\{a_n\}$, since l_1 is the **least** upper bound.

Therefore, $m < a_n$ for some n, and so m is not in the interval $I_n = [a_n, b_n]$.

Similarly, if $m > l_1$, $m > b_n$ for some n, and therefore there is an interval in which it does not lie. (In fact, once we have one such interval I_n , we know that it does not lie in any of the subsequent intervals I_{n+k} either.)

Therefore this common element is unique.

Comment. This property,

that a set of nested intervals contains a unique real number

is sometimes taken as the defining axiom for real numbers.

In this case, the least upper bound axiom is then deduced as a consequence.

Definition.

Given a set S of numbers, the number l is called a *limit point* or accumulation point of the set, if, given any $\epsilon > 0$ there is a number $x \neq l$ in S such that $|x-l| < \epsilon$.

The limit point may or may not be a member of S.

If we start with $\epsilon_1 > 0$ and determine such a number $x_1 \in S$, then we can define $\epsilon_2 = |x_1 - l|$.

Since $x_1 \neq l, \epsilon_2 > 0$, and we can determine a number $x_2 \neq l \in S$ such that

$$|x_2 - l| < |x_1 - l| < \epsilon_1$$
.

Note that $x_2 \neq x_1$.

Continuing in this way, we see that if l is a limit point, then for any $\epsilon > 0$ there are infinitely many points x in S such that

$$|x-l| < \epsilon \; .$$

Conversely, if every neighbourhood of l contains infinitely many points of S, it contains at least one point of S which differs from l. Therefore l is a limit point.

The Bolzano-Weierstrass Theorem.

A bounded infinite set of real numbers has a limit point.

Let S be our infinite set of numbers, and let a_1 be the greatest lower bound for S and b_1 be the least upper bound.

Divide the interval $[a_1, b_1]$ into two equal sections; $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$.

At least one of these sections contains infinitely many points of S.

If the first section contains infinitely many points of S, let $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$; otherwise let $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$.

In either case, $a_1 \leq a_2 < b_2 \leq b_1$, $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$ and there are infinitely many points of S in the interval $[a_2, b_2]$.

We proceed inductively to define a nested sequence of intervals $[a_n, b_n]$ of length

$$\frac{1}{2^{n-1}}(b_1-a_1)$$

each of which contains infinitely many points of S.

Since the length of these intervals goes to 0 as $n \to \infty$, there is a unique number l which lies in every interval.

Given any $\epsilon > 0$, we can find an integer N such that $b_N - a_N < \epsilon$. For every point x in this interval, $|x - l| \le b_N - a_N < \epsilon$, so that there are infinitely many points of S such that $|x - l| < \epsilon$. This shows that l is a limit point of S.

Note that a set may have more than one limit point.

For example. if

$$S = \left\{ (-1)^n \frac{n+1}{n} \right\}$$

then 1 and -1 are limit points of S,

while if S is the set of rational numbers between 0 and 1 then every real number in [0, 1] is a limit point!

We are now in a position to prove that a Cauchy sequence in \mathbb{R} converges.

In order to apply the previous result, we need to show that such a sequence constitutes a bounded set.

Suppose that $\{x_n\}$ is a sequence of real numbers with the property that, given any $\epsilon > 0$, we can find an integer N such that $|x_n - x_m| < \epsilon$ for every m, n > N. In particular, taking $\epsilon = 1$, $|x_n - x_m| < 1$ for all $m, n > N_1$ say.

Therefore $|x_n - x_{N_1+1}| < 1$ for every $n > N_1$, so that $|x_n| < 1 + |x_{N_1+1}|$ for every $n > N_1$, and the sequence is bounded above by the maximum, M, of the finite set of numbers

$$\{|x_1|, |x_2|, \ldots, |x_{N_1}|, 1+|x_{N_1+1}|\},\$$

and below by -M.

In order to apply the Bolzano-Weierstrass theorem we also need an infinite set. In practice the terms of a sequence are usually different. However we need to guard against occasional pathalogical examples.

If the sequence takes only a finite number of different values for $n > N_1$, then there must be some value l which is taken infinitely often.

But now, given any $\epsilon > 0$ there is an integer N such that $|x_n - x_m| < \epsilon$ for all n, m > N. Since l occurs infinitely often in the sequence, one at least of these x_m takes the value l, so that $|x_n - l| < \epsilon$ for all n > N, and the sequence converges to l.

Otherwise, the sequence represents a bounded infinite set in \mathbb{R} , so that, by the Bozano-Weierstrass Theorem, the sequence has a limit point l.

Given any $\epsilon > 0$, there are infinitely many points of the sequence such that $|x_m - l| < \frac{1}{2}\epsilon$, and there is an integer N such that $|x_n - x_m| < \frac{1}{2}\epsilon$ for all m, n > N. Since at least one of these x_m coincide, it follows that $|x_n - l| < \epsilon$ for all n > N, and the sequence converges to l.

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MATH 3402

TUTORIAL SHEET 1

1. Describe each of the following sets as the empty set, as $\mathbb{R},$ or in interval notation as appropriate:

(a)
$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right)$$

(b)
$$\bigcup_{n=1}^{\infty} (-n,n)$$

(c)
$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right)$$

(d)
$$\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 2 + \frac{1}{n} \right)$$

(e)
$$\bigcup_{n=1}^{\infty} \left(\mathbb{R} \setminus \left(-\frac{1}{n}, \frac{1}{n} \right) \right)$$

(f)
$$\bigcap_{n=1}^{\infty} \left(\mathbb{R} \setminus \left[\frac{1}{n}, 2 + \frac{1}{n} \right] \right)$$

2. Prove that if $\{a_n\}$ and $\{b_n\}$ are Cauchy, so are $\{a_n + b_n\}$ and $\{a_n b_n\}$.

3. Let a_0 and a_1 be distinct real numbers. Define $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for each positive integer $n \geq 2$. Show that $\{a_n\}$ is a Cauchy sequence.

4. Suppose x is an accumulation point of $\{a_n : n \in \mathbb{N}\}$. Show that there is a subsequence of $\{a_n\}$ that converges to x.

5. Let $\{a_n\}$ be a bounded sequence of real numbers. Prove that $\{a_n\}$ has a convergent subsequence.