Compactness and Completeness

We have seen that if S is a compact set (in a topological space), then every infinite subset of S has an accumulation point in S.

Note that this means that the interval $[0,1]$ considered as a subset of $\mathbb{Q}$ is closed and bounded in $\mathbb{Q}$ but is not compact.

The converse holds for metric spaces:
If S is a set in a metric space with the property that every infinite subset of S has an accumulation point in S, then S is compact.

To show this, we first show that if $S \subset (X,d)$ has this property, then there is a countable family $\mathcal{N}$ of open sets $N \subset X$ such that, if U is any open set in X and $x \in S \cap U$, there is some $N \in \mathcal{N}$ for which $x \in N \subset U$.

For each positive integer n we can find a finite set of points $x_1, \ldots, x_{f(n)}$ in S such that the sets $\{d(x,x_i) < \frac{1}{n}\}$ cover S.
If this were not so, there would be some n and a infinite sequence $\{x_i\}$ in S for which $d(x_i,x_j) \geq \frac{1}{n}$ for $i \neq j$. Since this sequence could not have a limit point in S, this contradicts the assertion about S.
The collection of all these open sets constitutes $\mathcal{N}$.

Now, for any $x \in S \cap U$, $x \in U$ which is open, so that there is some $\epsilon > 0$ such that $\{d(y,x) < \epsilon\} \subset U$.
Let $\frac{1}{n} < \frac{\epsilon}{2}$. The collection of open sets associated with this value of n cover S, therefore $x \in N$ for one of them, and $N \subset \{d(y,x) < \epsilon\} \subset U$.

Now suppose that $\mathcal{U}$ is an open covering of S.
We specify a subfamily $\mathcal{N}^*$ of $\mathcal{N}$ as follows:
$N \in \mathcal{N}^*$ iff $N \subset U$ for some $U \in \mathcal{U}$.
Since $\mathcal{U}$ covers S, every $x \in S$ is in $S \cap U$ for some U and therefore in some $N \in \mathcal{N}^*$, so that $\mathcal{N}^*$ covers S also.
Now, for each $N \in \mathcal{N}^*$ we choose one set U in $\mathcal{U}$ such that $N \subset U$. This subset $\mathcal{U}^*$ of $\mathcal{U}$ is countable and covers S.
Index the sets in $\mathcal{U}^*$ by $U_1, U_2, \ldots$.
If no finite subcollection of $\mathcal{U}^*$ covers S, then there must be a point x_n in $S \setminus (U_1 \cup \cdots \cup U_n)$ for each n.
This infinite set $\{x_n\}$ has a limit point $x_0 \in S$.
Therefore $x_0 \in U_N$ for some N.
Since x_0 is a limit point and U_N is open, infinitely many terms in the sequence $\{x_n\}$ lie in U_N.
In particular, for some $n > N$, $x_n \in U_N$, which is a contradiction.
Therefore some finite subcollection of $\mathcal{U}^*$, which is in turn a subcollection of $\mathcal{U}$ covers S, and S is compact.

Hence, in a metric space a set S is compact if and only if every infinite subset of S has an accumulation point in S.

Equivalently, in a metric space, a set S is compact if and only if every sequence in S contains a convergent subsequence with limit in S.

Completeness.

We have seen that if $\{x_n\}$ is a convergent sequence in the metric space (X, d), then given any $\epsilon > 0$ there is an integer N such that

$$d(x_m, x_n) < \epsilon \ \forall \ m, n > N \ .$$

This is Cauchy's Criterion.

A sequence which satisfies Cauchy's criterion, (without necessarily converging) is called a Cauchy sequence.

A metric space X with the property that every Cauchy sequence in X has a limit in X is said to be **complete**.

If the metric space (X, d) is not complete, and completeness is considered desirable, then there are two alternatives.

We can change either X or d.

The first process is called **completion**.

For example, this is the process by which the rational numbers are extended to become the real numbers.

Firstly, we define *isometry*.

Two metric spaces (X, d_X) and (Y, d_Y) are isometric if there is a one to one function f from X to Y such that

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

for every pair of points $x_1, x_2 \in X$.

The fundamental result is as follows.

Let (X, d_X) be an incomplete metric space.

There exists a complete metric space (Y, d_Y) and a subset Y_0 dense in Y such that X and Y_0 are isometric.

Let $\{x_n\}$ and $\{x'_n\}$ be Cauchy sequences in X.

We say that these sequences are *equivalent*, $\{x_n\} \sim \{x'_n\}$, if $d_X(x_n, x'_n) \to 0$ as $n \to \infty$.

This relation is reflexive, symmetric and transitive, and hence divides the set of all Cauchy sequences in X into equivalence classes.

Our space Y is the set of all these equivalence classes.

We define the metric d_Y on Y:

If $\{x_n\}$ and $\{x'_n\}$ are Cauchy sequences in X, then $d_X(x_n, x'_n)$ is a Cauchy sequence in $\mathbb{R}$.

Given any $\epsilon > 0$, there exist N_1 and N_2 such that $d_X(x_m, x_n) < \frac{\epsilon}{2}$ for all $m, n > N_1$, and $d_X(x'_m, x'_n) < \frac{\epsilon}{2}$ for all $m, n > N_2$.

Then for $m, n > \max(N_1, N_2)$,

$$\begin{aligned}
&|d_X(x_n, x'_n) - d_X(x_m, x'_m)| \\
&= |d_X(x_n, x'_n) - d_X(x_m, x'_n) + d_X(x_m, x'_n) - d_X(x_m, x'_m)| \\
&\leq |d_X(x_n, x'_n) - d_X(x_m, x'_n)| + |d_X(x_m, x'_n) - d_X(x_m, x'_m)| \\
&\leq d_X(x_n, x_m) + d_X(x'_n, x'_m) < \epsilon
\end{aligned}$$

Since $\mathbb{R}$ is complete, this sequence converges, and we set

$$d_Y(\{x_n\}, \{x'_n\}) = \lim_{n\to\infty} d_X(x_n, x'_n)$$

This limit is unchanged if we replace $\{x_n\}$ and $\{x'_n\}$ by equivalent series $\{a_n\}$ and $\{a'_n\}$, since

$$|d_X(x_n, x'_n) - d_X(a_n, a'_n)| \leq d_X(x_n, a_n) + d_X(x'_n, a'_n)$$

in the same way as above, and the right hand side goes to zero as $n \to \infty$.

Among the Cauchy sequences in X we have those for which $x_n = x$ for all n.

The equivalence classes to which these sequences belong form a subset Y_0 of Y, and for x, x' in X, the mapping from X to Y_0 which maps x onto $\{x\}$ is obviously one-one, and

$$d_Y(\{x\}, \{x'\}) = \lim_{n\to\infty} d_X(x, x') = d_X(x, x')$$

so that this mapping is an isometry.

Now consider $Cl(Y_0)$.

If $y \in Y$, there is a Cauchy sequence $\{x_n\}$ in X which defines y.

Given any $\epsilon > 0$, there is an integer N such that $d_X(x_n, x_N) < \frac{\epsilon}{2}$ for all $n > N$.

Let $y_0 \in Y_0$ be the class containing the constant sequence $\{x_N\}$. Then

$$d_Y(y, y_0) = \lim_{n\to\infty} d_X(x_n, x_N) \leq \frac{\epsilon}{2} < \epsilon$$

and $y \in Cl(Y_0)$. Therefore $Cl(Y_0) = Y$, and Y_0 is dense in Y.

Finally, we show that Y is complete.

Suppose that $\{y_n\}$ is a Cauchy sequence in Y.

For each n, choose a $z_n \in Y_0$ such that $d_Y(y_n, z_n) < \frac{1}{n}$. (This is possible since Y_0 is dense in Y.)

Now

$$\begin{aligned} d_Y(z_n, z_m) &\leq d_Y(z_n, y_n) + d_Y(y_n, y_m) + d_Y(y_m, z_m) \\ &< \frac{1}{n} + d_Y(y_n, y_m) + \frac{1}{m} \end{aligned}$$

so that $\{z_n\}$ is a Cauchy sequence in Y.

To each $z_n \in Y_0$, we can associate an element $x_n \in X$, and

$$d_X(x_n, x_m) = d_Y(z_n, z_m)$$

therefore $\{x_n\}$ is a Cauchy sequence in X, which defines an element $y \in Y$.

Now,

$$\begin{aligned} d_Y(y_n, y) &\leq d_Y(y_n, z_n) + d_Y(z_n, y) \\ &< \frac{1}{n} + d_Y(z_n, y) \\ &< \frac{1}{n} + \lim_{m\to\infty} d_X(x_n, x_m) \end{aligned}$$

so that $\{y_n\}$ converges to $y \in Y$, and Y is complete.

In practice we identify X and Y_0, and regard X as a dense set in Y.

Alternatively, we could adjoin the elements of $Y \setminus Y_0$ to X to form a space $\hat{X}$ isometric to Y in which X is a dense set.

This space is called the completion of X.