Contraction mappings
Applications and extensions

The contraction mapping theorem can also be used as a theoretical tool.

In particular, it can be used to prove the existence and uniqueness of the solution of the initial value problem

$$y' = f(x, y) \ ; \ y(a) = b$$

under certain conditions on the function f.

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on the rectangle

$$\mathcal{R} = [a - h, a + h] \times [b - k, b + k]$$

and on this rectangle satisfies the additional (Lipschitz) condition that there is a constant M such that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in \mathcal{R}$.
(In particular, this condition will be satisfied if $\partial f/\partial y$ exists and is bounded in $\mathcal{R}$.)
Then there is an $\alpha > 0$ such that the initial value problem has a unique solution for $a - \alpha \leq x \leq a + \alpha$.

We begin by modifying $\mathcal{R}$ if necessary to ensure that any solution remains bounded by $b - k$ and $b + k$.
Since $\mathcal{R}$ is closed and bounded in $\mathbb{R}^2$, it is compact.
Since f is continuous on $\mathcal{R}$, its range is compact in $\mathbb{R}$.
In particular, it is bounded; there exists a constant K such that

$$|f(x, y)| \leq K \ \forall \ x, y \in \mathcal{R} \ .$$

Therefore, any solution in $\mathcal{R}$ is bounded by the lines

$$y - b = \pm K(x - a) \ .$$

If $k \geq Kh$, this region is bounded by $b - k$ and $b + k$ for $x \in [a - h, a + h]$.
Otherwise, set $h' = k/K$, and reduce the region $\mathcal{R}$ appropriately;

$$\mathcal{R}' = [a - h', a + h'] \times [b - k, b + k] \ .$$

Since $\mathcal{R}' \subset \mathcal{R}$, the conditions on f are unaffected by this change.

Next we replace the differential equation by an integral equation:
If $y(x)$ is a solution of the differential equation, then y is differentiable.
Therefore y is continuous, and $f(x, y(x))$ is a continuous function of x. Hence it is integrable, and

$$\int_a^x y'(t)\,dt = \int_a^x f(t, y(t))\,dt$$

$$y(x) = b + \int_a^x f(t, y(t))\,dt \ .$$

Conversely, if $y(x)$ is a solution of this integral equation, then $y(x)$ is continuous since the right-hand side of the equation is continuous.

Therefore the integrand $f(t, y(t))$ is continuous, and by the fundamental theorem of the calculus

$$\int_a^x f(t, y(t))\, dt$$

is differentiable, and

$$\frac{d}{dx}\int_a^x f(t, y(t))\, dt = f(x, y(x))$$

Hence y is a solution of the initial value problem also.

We now need to show that the mapping

$$\mathcal{F}(\phi) = b + \int_a^x f(t, \phi(t))\, dt$$

derived from the integral equation is a contraction mapping on an appropriate complete metric space. If it is, then the fixed point y of the mapping is the unique solution of the integral equation.

We choose the space $C^*(a-\alpha, a+\alpha)$ of functions continuous on $[a-\alpha, a+\alpha]$ and bounded by $b-k$ and $b+k$, together with the uniform sup metric;

$$||f-g|| = \max_{x\in[a-\alpha,a+\alpha]} |f(x)-g(x)| \ ,$$

where $0 < \alpha \le h'$ will be chosen to ensure that the mapping is a contraction. This is a closed subset of a known complete metric space, and is therefore complete.

Note that the condition $|x-a| \le h'$ ensures that

$$\begin{aligned} |\mathcal{F}(\phi) - b| &= \left|\int_a^x f(t, \phi(t))\, dt\right| \\ &\le \left|\int_a^x |f(t, \phi(t))|\, dt\right| \\ &\le K|x-a| \le Kh' \le k \end{aligned}$$

so that $\mathcal{F}(C^*) \subset C^*$.

Specifically, we choose α so that $\alpha M < 1$, where M is the Lipschitz constant.

If ϕ and ψ are two elements in C^*, then

$$\begin{aligned} ||\mathcal{F}(\phi) - \mathcal{F}(\psi)|| &= \ \sup\left|\int_a^x (f(t,\phi(t)) - f(t,\psi(t)))\, dt\right| \\ \left|\int_a^x (f(t,\phi(t)) - f(t,\psi(t)))\, dt\right| &\le \left|\int_a^x |f(t,\phi(t)) - f(t,\psi(t))|\, dt\right| \\ &\le \left|\int_a^x M|\phi-\psi|\, dt\right| \\ &\le \left|\int_a^x M||\phi-\psi||\, dt\right| \\ &\le M\alpha||\phi-\psi|| \\ ||\mathcal{F}(\phi) - \mathcal{F}(\psi)|| &\le M\alpha||\phi-\psi|| \end{aligned}$$

Hence $\mathcal{F}$ is a contraction mapping, and the result follows.

e.g. Consider the differential equation

$$y' = y^2 + x \ ; \ y(0) = 0$$

in the square $\mathcal{R} = [-1, 1] \times [-1, 1]$.

The function $f(x, y) = y^2 + x$ is obviously continuous in $\mathcal{R}$, and

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |y_1^2 - y_2^2| \\ &= |y_1 - y_2|\,|y_1 + y_2| \leq 2|y_1 - y_2| \end{aligned}$$

for $(x, y_1), (x, y_2) \in \mathcal{R}$.

On $\mathcal{R}$, the maximum of $|y^2 + x|$ is 2, so that we have to replace $\mathcal{R}$ by $\mathcal{R}' = [-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]$ to ensure that we map C^* into C^*.

Finally, for $M\alpha < 1$ we need to choose $\alpha < \frac{1}{2}$.

The contraction mapping theorem now guarantees a unique solution on $[-\alpha, \alpha]$ by iterating

$$y(x) = \int_0^x (y^2(t) + t)\, dt \ .$$

We can approximate this solution by choosing $\phi_0(x) = y(0) = 0$, giving

$$\begin{aligned} \phi_1(x) &= \frac{1}{2}x^2 \\ \phi_2(x) &= \frac{1}{2}x^2 + \frac{1}{20}x^5 \\ \phi_3(x) &= \frac{1}{2}x^2 + \frac{1}{20}x^5 + \frac{1}{160}x^8 + \frac{1}{4400}x^{11} \end{aligned}$$

The power series generated by this process in fact converges to the solution in a larger interval than $[-\alpha, \alpha]$.

The solution of the equation can be expressed in the form $y = -u'/u$, where

$$u(x) = 1 - \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} - \frac{x^9}{2.3.5.6.8.9} + \dots$$

u has a zero for $x \sim 2$, which generates a vertical asymptote for y.

This result can be extended to systems of differential equations, where $y \in \mathbb{R}^n$, and from there to n^{th} order initial value problems.

Extensions.

While $\cos x$ is not a contraction mapping on $\mathbb{R}$, the iterated function $\cos(\cos x)$ satisfies

$$\left| \frac{d}{dx}(\cos(\cos x)) \right| = |\sin(\cos x)\sin(x)| < \sin 1 < 1$$

and is a contraction on $\mathbb{R}$.

Similarly, $\exp(-x)$ is not a contraction on $\mathbb{R}$, but $\exp(-\exp(-x))$ satisfies

$$\left| \frac{d}{dx}(\exp(-\exp(-x))) \right| = |\exp(-x - \exp(-x))| \leq e^{-1} < 1$$

and is a contraction on $\mathbb{R}$.

Consider then the situation in which $f : A \to A$ is not a contraction mapping but f^N, where the superscript represents the N^{th} iteration of the function, is a contraction.

For any $x \in A$, the sequence $x, f^N(x), f^{2N}(x), \dots$ converges to the fixed point of f^N.

Now consider a sequence $\{f^n(x_0)\}$ of iterates of f.

We can write this sequence as N subsequences

$$\begin{array}{cccc} x_0 & x_N = f^N(x_0) & x_{2N} = f^{2N}(x_0) & \dots \\ x_1 = f(x_0) & x_{N+1} = f^N(x_1) & x_{2N+1} = f^{2N}(x_1) & \dots \\ . & . & . & \dots \\ x_{N-1} = f^{N-1}(x_0) & x_{2N-1} = f^N(x_{N-1}) & x_{3N-1} = f^{2N}(x_{N-1}) & \dots \end{array}$$

each of which converges to the fixed point of f^N. Therefore the whole sequence also converges to this fixed point.

Furthermore, if $f^N(a) = a$, $f^{N+1}(a) = f^N(f(a)) = f(a)$, and since the fixed point is unique, $f(a) = a$.

Therefore, f has a unique fixed point in A, and the sequence converges to it.

For instance, with $f(x) = e^{-x}$ and $x_0 = 1$, we obtain

$$\begin{aligned} x_2 &= 0.36788 \\ x_3 &= 0.69220 \\ x_4 &= 0.50047 \\ x_5 &= 0.60624 \\ x_6 &= 0.54540 \\ x_7 &= 0.57961 \\ x_8 &= 0.56012 \\ &\dots \\ a &= 0.56714\dots \end{aligned}$$

As another example, consider the space $C(0, \frac{\pi}{2})$ of continuous real functions on $[0, \frac{\pi}{2}]$ together with the uniform sup metric.

$$\mathcal{F}(f) = \int_0^x (f(t) + t) \sin t \, dt$$

maps C into C.

However, if $\phi(x) = -x$ and $\psi(x) = 1 - x$,

$$||\phi - \psi|| = \sup_{0 \le x \le \pi/2} |1| = 1$$

, while

$$\begin{aligned} \mathcal{F}(\phi) &= 0 \\ \mathcal{F}(\psi) &= \int_0^x \sin t \, dt = 1 - \cos x \\ ||\mathcal{F}(\phi) - \mathcal{F}(\psi)|| &= \sup_{0 \le x \le \pi/2} |1 - \cos x| = 1 \end{aligned}$$

so that $\mathcal{F}$ is not a contraction mapping on C.

On the other hand, in general

$$\begin{aligned}
|\mathcal{F}(\phi)-\mathcal{F}(\psi)| &= \left|\int_0^x (\phi(t)-\psi(t))\sin t\,dt\right| \\
&\le \int_0^x |\phi(t)-\psi(t)|\,dt \\
&\le x||\phi-\psi|| \\
\left|\mathcal{F}^2(\phi)-\mathcal{F}^2(\psi)\right| &\le \int_0^x |\mathcal{F}(\phi)-\mathcal{F}(\psi)|\,dt \\
&\le \int_0^x x||\phi-\psi||\,dt \\
&\le \frac{1}{2}x^2||\phi-\psi|| \\
\left|\mathcal{F}^3(\phi)-\mathcal{F}^3(\psi)\right| &\le \int_0^x \left|\mathcal{F}^2(\phi)-\mathcal{F}^2(\psi)\right|\,dt \\
&\le \int_0^x \frac{1}{2}x^2||\phi-\psi||\,dt \\
&\le \frac{1}{6}x^3||\phi-\psi|| \\
&\le \frac{1}{6}\left(\frac{\pi}{2}\right)^3||\phi-\psi|| \\
&< 0.65||\phi-\psi||
\end{aligned}$$

so that $\mathcal{F}^3$ is a contraction on C, and there is a unique continuous solution of

$$y(x) = \int_0^x (y(t)+t)\sin(t)\,dt$$

on $[0, \frac{\pi}{2}]$, namely

$$y(x) = \int_0^x t\sin t\exp(\cos t-\cos x)\,dt\ .$$

Differential equations revisited

The solution given above was found by solving the equivalent differential equation

$$y' = (y+x)\sin x\ ;\ y(0)=0\ .$$

When we derived the existence and uniqueness proof for differential equations earlier, we had to restrict our iteration to an interval $[a-\alpha, a+\alpha]$ where $\alpha M < 1$ in order to ensure that we had a contraction mapping.

The above example shows that we could achieve the same result by ensuring that $\mathcal{F}^N$ is a contraction mapping for some integer N.

This enables us to retain the interval $[a-h', a+h']$.

In the same fashion as above, we can show

$$|\mathcal{F}(\phi)-\mathcal{F}(\psi)| \le \left|\int_a^x M||\phi-\psi||\,dt\right| \le M|x-a|||\phi-\psi||$$

$$\left|\mathcal{F}^2(\phi)-\mathcal{F}^2(\psi)\right| \le \left|\int_a^x M||\mathcal{F}(\phi)-\mathcal{F}(\psi)||\,dt\right| \le \frac{1}{2!}M^2|x-a|^2||\phi-\psi||$$

$$|\mathcal{F}^n(\phi)-\mathcal{F}^n(\psi)| \le \left|\int_a^x M||\mathcal{F}^{n-1}(\phi)-\mathcal{F}^{n-1}(\psi)||\,dt\right| \le \frac{1}{n!}M^n|x-a|^n||\phi-\psi||$$

so that

$$||\mathcal{F}^n(\phi) - \mathcal{F}^n(\psi)|| \leq \frac{1}{n!}(Mh')^n ||\phi - \psi||$$

Since the power series for e^x converges for all x, there is some N such that

$$\frac{1}{N!}(Mh')^N < 1$$

and so we eventually have a contraction mapping as required.

Contraction mapping on compact sets

On the tutorial sheet you have seen that the condition

$$d(f(x), f(y)) < d(x, y) \ \forall \ x \neq y \in A$$

is sufficient for uniqueness, and that a continuous real function from $[a, b]$ to $[a, b]$ has at least one fixed point.

We can combine and generalize slightly these results to give:

If (X, d) is a compact metric space, and $f : X \to X$ satisfies

$$d(f(x), f(y)) < d(x, y) \ \forall \ x \neq y \in X \ ,$$

then f has a unique fixed point in X.

Furthermore, for any $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to the fixed point.

For every $x \in X$, consider the function $F : X \to \mathbb{R}$ given by

$$F(x) = d(x, f(x)) \ .$$

The set $F(X) \subset \mathbb{R}$ is bounded below by 0, so it has a greatest lower bound, α, say.

Suppose that there is no element $a \in X$ such that $F(a) = \alpha$.

Taking $\epsilon_1 = 1$, there is an element $x_1 \in X$ such that

$$\alpha < F(x_1) < \alpha + \epsilon_1 \ .$$

Now take $\epsilon_2 = \frac{1}{2}(F(x_1) - \alpha) > 0$. There is an element $x_2 \in X$ such that

$$\alpha < F(x_2) < \alpha + \epsilon_2 < \alpha + \frac{1}{2} \ .$$

Continuing in this way we can construct an infinite collection of points $x_n \in X$ such that

$$\alpha < F(x_n) < \alpha + \frac{1}{2^{n-1}} \ .$$

Since X is compact, this collection has a convergent subsequence whose limit a is in X, and for which

$$\alpha \leq F(a) < \alpha + \epsilon$$

for any $\epsilon > 0$.

Therefore $F(a) = \alpha$.

But now

$$F(f(a)) = d(f(a), f(f(a))) < d(a, f(a)) = F(a)$$

unless $a = f(a)$.

Therefore a is a fixed point of f, and the property

$$d(f(x), f(y)) < d(x, y) \ \forall \ x \neq y \in X$$

ensures that it is unique.

Now consider an arbitrary $x_0 \in X$.

The sequence $\{x_n = f^n(x_0)\}$ has the property that

$$d(x_{n+1}, a) = d(f(x_n), f(a)) < d(x_n, a)$$

so that $\{d(x_n, a)\}$ is a monotonic decreasing sequence in $\mathbb{R}$ which is bounded below by 0. Therefore it has a limit $\beta \geq 0$.

Now the set $\{x_n\}$ is an infinite collection in X which has a convergent subsequence in X, which has a limit y in X, for which $d(y, a) = \beta$.

Suppose $\{x_{n_i}\} \to y$.

Since f is uniformly continuous on X, $\{f(x_{n_i})\} \to f(y)$. That is: the sequence $\{x_{n_i+1}\}$ is another convergent subsequence of $\{x_n\}$. Therefore $d(f(y), a) = \beta$ also.

But

$$d(f(y), a) = d(f(y), f(a)) < d(y, a)$$

unless $y = a$, which implies that $\beta = 0$.

In turn this gives $d(x_n, a) \to 0$ as $n \to \infty$, and $\{x_n\}$ converges to a as required.

The implicit function theorem

Let $F : \mathcal{R} = \mathbb{R} \times [a, b] \to \mathbb{R}$ be continuous on $\mathcal{R}$, and differentiable with respect to its first argument.

If there are numbers m and M such that

$$0 < m \leq \frac{\partial F(x,t)}{\partial x} \leq M \ \forall \ (x,t) \in \mathcal{R}$$

then there is a unique continuous function $x : [a, b] \to \mathbb{R}$ such that $F(x(t), t) = 0$ for all $t \in [a, b]$.

As Usual, we choose for our complete metric space the set $C(a, b)$ of functions continuous on $[a, b]$ together with the uniform metric.

We construct a contraction mapping for this problem in a fashion analogous to the earlier numerical examples:

$$\mathcal{F}(\phi(t)) = \phi(t) - \frac{1}{M} F(\phi(t), t) \ .$$

This obviously maps C into C.

For $\phi(t), \psi(t) \in C$, then for each $t' \in [a, b]$, the Mean Value Theorem applied to the first variable in F gives

$$F(\phi(t'), t') - F(\psi(t'), t') = F_x(c, t')(\phi(t') - \psi(t'))$$

for some c between $\phi(t')$ and $\psi(t')$.

Therefore

$$\begin{aligned}
&\mathcal{F}(\phi(t')) - \mathcal{F}(\psi(t')) \\
&= \phi(t') - \frac{1}{M} F(\phi(t'), t') - \psi(t') + \frac{1}{M} F(\psi(t'), t') \\
&= (\phi(t') - \psi(t')) \left(1 - \frac{1}{M} F_x(c, t')\right) \\
|\mathcal{F}(\phi(t')) - \mathcal{F}(\psi(t'))| &\leq \left(1 - \frac{m}{M}\right) |\phi(t') - \psi(t')| \leq k||\phi - \psi|| \\
||\mathcal{F}(\phi) - \mathcal{F}(\psi)|| &\leq k||\phi - \psi||
\end{aligned}$$

where $k = (M - m)/M < 1$.

Hence $\mathcal{F}$ is a contraction mapping on C, and the result follows.