1 a) $w = \sinh^{-1} z \iff z = \sinh w = \dfrac{e^{w} - e^{-w}}{2}$

So $2z = e^{w} - e^{-w} \overset{\cdot e^{w}}{\Rightarrow} 2ze^{w} = e^{2w} - 1$

i.e. $e^{2w} - 2ze^{w} - 1 = 0$

$$\Rightarrow e^{w} = \frac{2z + (4z^2+4)^{1/2}}{2}$$

$$= z + (z^2+1)^{1/2}$$

So $w = \sinh^{-1} z = \log e^{w} = \log\left(z + (z^2+1)^{1/2}\right)$

as req^{d}.

b) $\sinh z = 4i \iff z = \sinh^{-1} 4i$

$$\overset{a)}{=} \log\left(4i + (-16+1)^{1/2}\right)$$

$$= \log(4i \pm \sqrt{15}\, i)$$

$$= \log\left[(4+\sqrt{15})i\right], \log\left[(4-\sqrt{15})i\right]$$

$$= \left\{\ln|4+\sqrt{15}| + (2n+\tfrac{1}{2})\pi i,\ n \in \mathbb{Z}\right\} \cup$$

$$\left\{\ln|4-\sqrt{15}| + (2n+\tfrac{1}{2})\pi i,\ n \in \mathbb{Z}\right\}.$$

② (i) $f(z) = u(x,y) + iv(x,y)$ for $z = x+iy$,

where $u = x^2+y^2$, $v = -2xy$

So $\quad u_x = 2x \quad u_y = 2y$

$\quad\quad v_x = -2y \quad v_y = -2x$

$C/R_{I} \Rightarrow \quad u_x = v_y \Rightarrow 2x = -2x \Leftrightarrow x = 0.$

$C/R_{II} \Rightarrow \quad u_y = -v_x \Rightarrow 2y = +2y$, which is always true.

So C/R hold only on the line $\{x=0\}$, i.e., the imaginary axis.

(ii) By (i), f is only possibly differentiable on the Imaginary axis (since C/R are necessary for diff^{bility}). Since u, v, u_x, v_x, u_y, v_y are defined & cts on $\mathbb{R}^2$, by (i) f is differentiable precisely on the Im axis.

(iii) f is not differentiable on any nbhd of any pt. in $\mathbb{C}$, and hence is nowhere analytic.

(iv) analytic $\Rightarrow$ diff^{ble} but not vice versa, so no contradiction.

(3) a) $\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0$

So for $f(z) = \frac{z^9 + 7z}{z^5 + 17z}$, RHS $=$

$$\lim_{z \to 0} 1 \Big/ \frac{\frac{1}{z^9} + 7 \cdot \frac{1}{z}}{\frac{1}{z^5} + 17/z} = \lim_{z \to 0} \frac{1/z^5 + 17/z}{1/z^9 + 7/z}$$

$$= \lim_{z \to 0} \frac{z^4 + 17z^8}{1 + 7z^8} = 0 \text{ as req}^{d}.$$

③ b) Put $f(z) = \dfrac{az+b}{cz+d}$.

$f(\infty) = 0 \Rightarrow a = 0$ ①

$f(0) = -i \Rightarrow b/d = -i$ ②

$f(1) = \infty \Rightarrow c = -d$. ③

So put $d = 1$: ② $\Rightarrow b = -i$, ③ $\Rightarrow c = -1$

So $f(z) = \dfrac{-i}{-z+1} = \dfrac{i}{z-1}$.

(4) a) $(1+i)^z = \exp(z \log(1+i))$, where we need to specify a single-valued f.ⁿ to differentiate, i.e., specify a branch of log, e.g. Log. Then for this branch, specifically for $-\pi < \arg z < \pi$ $(z \neq 0)$, there holds:

$$\frac{d}{dz}\left[(1+i)^z\right] = \mathrm{Log}(1+i)\exp(z\,\mathrm{Log}(1+i))$$

$$= \mathrm{Log}(1+i)(1+i)^z$$

$$= (\ln\sqrt{2} + i\pi/4)(1+i)^z$$

$$= (\tfrac{1}{2}\ln 2 + i\pi/4)(1+i)^z .$$

4 b) $\sin z = \cosh 4$: (1) for $z = x + iy$,

$$\sin z = \sin x \cosh y + i \cos x \sinh y \quad (2)$$

So equating real and imaginary parts of (1) & (2):

$$\left.\begin{array}{l}\sin x \cosh y = \cosh 4 \\ \cos x \sinh y = 0\end{array}\right\} \begin{array}{l}(\alpha)\\(\beta)\end{array}$$

$(\beta) \Rightarrow x = \pm\frac{\pi}{2} + 2n\pi \quad n \in \mathbb{Z} \quad (\gamma)_\pm$

OR $y = 0 \quad (\delta)$

For $(\gamma)_+$: $(\alpha) \Rightarrow \cosh y = \cosh 4$, so $y = \pm 4$;

For (δ), $\cosh y = 1$, so $(\alpha) \Rightarrow \sin x = \cosh 4 > 1$, which is not possible.

For $(\gamma)_-$: $-\cosh y = \cosh 4$, which has no sol^n.

Hence the only sol^{n}s are

$$\left\{\frac{\pi}{2} + 2n\pi \pm 4i, \; n \in \mathbb{Z}\right\}.$$