MATH3401/7431 mid-sem 2022, sol^{n}s.

Q1.

(a) $z^2 - 3iz - 2 = 0 \Leftrightarrow (z-2i)(z-i) = 0$
$\Leftrightarrow z = 2i, i$ (or use quadratic formula).

b) For $\xi = z^2$, we have
$\xi^2 - 3i\xi - 2 = 0$, so from (i),
$\xi = 2i$ ⓐ or
$\xi = i$ ⓑ
For ⓐ, $z^2 = 2i \Rightarrow z^2 = 2e^{i\pi/2}$
$\Rightarrow z = \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{5\pi i/4}$
$= 1+i, -1-i$
For ⓑ, $z^2 = i = e^{i\pi/2}$
$\Rightarrow z = e^{i\pi/4}, e^{5\pi i/4} = \frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}$.
Hence, all solutions are
$\left\{ 1+i, -1-i, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right\}$.

2 a) Put $w = \tan z$.
Want to solve $z = \tan w = \dfrac{\sin w}{\cos w}$ for w, where $\cos w \neq 0$.

So: $z = \dfrac{\sin w}{\cos w} = \dfrac{(e^{iw} - e^{-iw})/2i}{(e^{iw} + e^{-iw})/2} = \dfrac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})}$

$$= \frac{e^{2iw} - 1}{i(e^{2iw} + 1)}$$

$\Leftrightarrow zi(e^{2iw} + 1) = e^{2iw} - 1$

$\Leftrightarrow e^{2iw}(zi - 1) = -(zi + 1)$ (*)

If $zi = 1$, i.e. $z = -i$, (*) means $0 = -2$, which has no solⁿ.
Similarly if $zi = -1$, i.e. $z = i$, (*) means $-2e^{2iw} = 0$, which again has no solⁿ.
Hence for $z \neq \pm i$, (*) $\Rightarrow e^{2iw} = \dfrac{-zi - 1}{zi - 1} = \dfrac{1 + zi}{1 - zi} = \dfrac{i - z}{i + z}$.

$\Leftrightarrow 2iw = \operatorname{Log} \dfrac{i - z}{i + z}$ on taking logs,

i.e. $w = \dfrac{1}{2i} \log \dfrac{i - z}{i + z} = \dfrac{-i}{2} \log \dfrac{i - z}{i + z} = \dfrac{i}{2} \operatorname{Log} \dfrac{i + z}{i - z}$.

b) $\tan z = 1 \Leftrightarrow z = \tan^{-1} 1 = \dfrac{i}{2} \log \dfrac{i + 1}{i - 1} = \dfrac{i}{2} \log(-i)$

$$= \frac{i}{2}\left[\ln|-i| + i \arg(-i)\right]$$

$$= \frac{i}{2}\left[0 + i(2k - \tfrac{1}{2})\pi\right] \quad k \in \mathbb{Z}$$

$$= (k + \tfrac{1}{4})\pi \quad k \in \mathbb{Z}$$

Q3.

a) (i) C/R : Since $u = x^2$, $v = y^3$, we have
$u_x = 2x$, $v_y = 3y^2$, $v_x = u_y = 0$.

So C/R : $\begin{cases} u_x = v_y \Rightarrow 2x = 3y^2 \\ v_x = -u_y \Rightarrow 0 = 0 \end{cases}$

$\Rightarrow$ C/R solved precisely on the curve $x = \frac{3}{2}y^2$.

(ii) u, v & all partials are cts on $\mathbb{C}$, so via "suff cond^{n}s" th^m, f is diff^{ble} on the curve $x = \frac{3}{2}y^2$.

(iii) f is not diff^{ble} on any nbhd of any pt in $\mathbb{C}$, so is nowhere analytic.

(iv) diff^{ble} $\underset{\Leftarrow}{\not\Rightarrow}$ analytic.

b) $(1+i)^z = \exp(z \log(1+i))$

In order to differentiate this using the chain rule, we need to specify a single value of $\log(1+i)$, e.g. $\mathrm{Log}(1+i)$.

Then, via the chain rule,

$$\frac{d}{dz}(1+i)^z = \mathrm{Log}(1+i)\exp(z\,\mathrm{Log}(1+i))$$
$$= \mathrm{Log}(1+i)\,(1+i)^z.$$

Q4.

a) Put $f(z) = \dfrac{az+b}{cz+d}$

$f(2) = 0 \Rightarrow 2a + b = 0$ ①

$f(i) = \infty \Rightarrow ic + d = 0$ ②

$f(0) = -2i \Rightarrow b/d = -2i$ ③

③ $\Rightarrow b = -2id$. Sub in ① $\Rightarrow 2a - 2id = 0$

$\div 2 \Rightarrow a - id = 0$ ④

$-i \cdot$ ② $\Rightarrow c - id = 0$ ⑤

④ & ⑤ $\Rightarrow a = c$. Set $a = c = 1$:

① $\Rightarrow b = -2$, ② $\Rightarrow d = -i$

$$\Rightarrow f(z) = \frac{z-2}{z-i}.$$

b) No such Möbius transformation exists, as we are requiring $5i$ and $1+i$ both to be mapped to i & Möbius transformations are 1-1.