1. (a) \( u(x, y) = \sqrt{|x^2y^3|}, \quad \nabla (u) = 0. \)

\[ 2. \quad \Rightarrow \nabla u = (0, 0) = 0. \]

\[ u_x(0,0) = \lim_{{h \to 0}} \frac{u(h,0) - u(0,0)}{h} = 0. \]

\[ u_y(0,0) = \lim_{{h \to 0}} \frac{u(0,h) - u(0,0)}{h} = 0. \]

So C/R hold at \((0,0)\).

(b) Consider \( \Delta z = h(1+i) \).

Then \[ \frac{f((0+\Delta z) - f(0)}{\Delta z} = \frac{\sqrt{|1h(1+i)|}}{h(1+i)} \]

\[ = \frac{|1h|}{h} \cdot \frac{1}{1+i} \quad \text{does not approach a limit as } h \to 0 (\Rightarrow \frac{1}{1+i} \text{ as } h \to 0^+, \triangleq \frac{-1}{1+i} \text{ as } h \to 0^-). \]

Hence \( f'(0) \) can't exist.

(c) C/R is necessary but not sufficient for differentiability, so no contradiction.
2) 

a) \[ \text{on } C_1: z(t) = -4 - it, \ 0 \leq t \leq 4. \] 
\[ \Rightarrow z'(t) = -i \] 
\[ \Rightarrow \int_C \text{Re}(z) \, dz = \int_0^4 (-4 + it) \, dt = 16i. \] 
\[ \partial z = 1 \] 

m C_2: \[ z = -4 - 4it + t, \ 0 \leq t \leq 4 \] 
\[ \Rightarrow \int_{C_2} \text{Re}(z) \, dz = \int_{-4}^{4-4i} \, dt = \frac{t^2}{2} \bigg|_{t=4} = 0. \] 

m C_3: \[ z = 4 - 4it + it, \ 0 \leq t \leq 4 \] 
\[ \Rightarrow \int_{C_3} \text{Re}(z) \, dz = \int_0^4 4 \, dt = 16i, \] 
\[ \Rightarrow \int_C \text{Re}(z) \, dz = \int_{C_1} + \int_{C_2} + \int_{C_3} = 16i + 0 + 16i = 32i. \]

b) \[ \text{on C: } z = 4e^{i\theta}, \ -\pi \leq \theta \leq 0, \ z' = 4ie^{i\theta} \] 
\[ \Rightarrow \int_C \text{Re}(z) \, dz = \int_0^{-\pi} \text{Re}(4e^{i\theta}) \, d\theta = 16 \int_0^{-\pi} \cos \theta e^{i\theta} \, d\theta \] 
\[ = 16i \int_0^{-\pi} \cos^2 \theta \, d\theta - 16i \int_0^{-\pi} \cos \theta \sin \theta \, d\theta \] 
\[ = 8i \int_0^{-\pi} (1 + \cos 2\theta) \, d\theta - 16i \left. \frac{\sin^2 \theta}{2} \right|_0^{-\pi} \] 
\[ = 8i \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{-\pi} - 0 = -8\pi i. \]

c) \[ \text{on C: } z = 4e^{-i\theta}, \ -\pi \leq \theta \leq 0, \ z' = -4ie^{-i\theta} \] 
\[ \Rightarrow \int_C \text{Re}(z) \, dz = -16i \int_0^{-\pi} \cos \theta e^{-i\theta} \, d\theta \] 
\[ = -8i \int_0^{-\pi} \cos^2 \theta \, d\theta = -8i \int_0^{-\pi} \cos \theta \sin \theta \, d\theta \] 
\[ \Rightarrow \text{of b): } z \rightarrow -8\pi i. \]

d) \[ z \rightarrow \text{Re}(z) \text{ is not analytic on any domain containing all of these curves, as the integrals depend on the path.} \]
3. a) The integrand is analytic on \( C \), with primitive \( e^z \log z \), where \( \log z \) is a branch of the logarithm chosen with \( \theta \) the branch cut on the positive imaginary axis, i.e.,
\[
\log(re^{i\theta}) = \ln r + i\theta, \quad \frac{\pi}{2} < \theta < \frac{5\pi}{2}.
\]
So, \( \int_{C} e^{\frac{z}{2}} \log z \, dz = e^{\frac{z}{2}} \log z \bigg|_{1}^{11} = \frac{1}{11} - \frac{1}{2} \pi i e + 2\pi i = e - \frac{1}{2} + \pi i. \)

b) Primitive is \( \sinh z \):
\[
\int_{C} \cosh z \, dz = \sinh z \bigg|_{11}^{11} = 0.
\]

4. a) Put \( f(z) = \frac{z^2 + 4z + 7}{(z^2 + 4)(z^2 + 2z + 2)} \)
\[
|f(z)| = \frac{1 + \frac{1}{2} z^2 + 3z^2}{(1 + \frac{1}{2} z^2)(\frac{1}{2} + 2z + 2)} \rightarrow \frac{1}{2}z^2 \quad \text{as} \quad |z| \rightarrow \infty.
\]
In particular, on \( C_r \), \( |f(z)| < \frac{1}{2} R^2 \) for \( R \) sufficiently large.

Since \( \text{Length}(C_r) = 2\pi R \)
\[
\text{So,} \quad |\int_{C_r} f(z) \, dz| \leq L R M_{f} = \frac{4\pi R}{2} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.
\]

b) Zeros of the denominator of \( f \) are \( \pm 2i, -1 \pm i \).
They all lie inside \( C \). So \( f \) is analytic on \( C \), and the region between them (if \( R \) large).
So \( \int_{C} f(z) \, dz = \int_{C_2} f(z) \, dz \). Since the LHS is independent of \( R \), \( f \) the RHS tends to zero as \( R \rightarrow \infty \). Then LHS (and indeed the RHS) must be identically zero.
\[ f(x) = \sin x \text{ is analytic on } C. \text{ So by Cauchy's formula:} \]
\[ f^{(6)}(x) = \frac{6!}{2\pi i} \int_C \frac{\sin z}{(z^6+1)^2} \, dz \]
\[ = 2\pi i \frac{d^6}{dx^6} \left. \sin(x) \right|_{x=-1} \]
\[ = \frac{2\pi i}{6!} (-\sin(-1)) \]
\[ = \frac{2\pi i (\sin 1)}{6!} \left( \approx 0.0034 \right) \]