COMPLEX ANALYSIS ASST 3

① a) $$\lim_{z\to\infty} \frac{4z^5}{z^5-42z} = \lim_{z\to 0} \frac{4\left(\frac{1}{z}\right)^5}{\left(\frac{1}{z}\right)^5 - 42\left(\frac{1}{z}\right)}$$

$$= \lim_{z\to 0} \frac{4}{1-42z^4} = 4.$$

b) $$\lim_{z\to\infty} \frac{z^4}{z^2+42z} = \infty \iff \lim_{z\to 0}\left[\frac{1/z^4}{1/z^2 + 42/z}\right]^{-1} = 0$$

$$\iff \lim_{z\to 0}\left[\frac{1}{z^2+42z^3}\right]^{-1} = 0 \iff \lim_{z\to 0}\left(z^2+42z^3\right) = 0,$$

which is true.

c) $$\lim_{z\to\infty} \frac{(az+b)^3}{(cz+d)^3} = \lim_{z\to 0} \frac{(a/z+b)^3}{(c/z+d)^3}$$

$$= \lim_{z\to 0}\left(\frac{a+bz}{c+dz}\right)^3 = a^3/c^3.$$

② a) real & imaginary parts ($u=2xy$, $v=x^2+y^2$) are defined on all of $\mathbb{C}$. Hence the function is defined on all of $\mathbb{C}$.

There holds: $u_x=2y$, $u_y=2x$, $v_x=2x$, $v_y=2y$.
So $C/R_I \Rightarrow 2y=2y$, which holds on $\mathbb{C}$, but

$C/R_{II} \Rightarrow u_y = -v_x \Rightarrow 2x = -2x$, which is only true on the imaginary axis. There is hence no point in $\mathbb{C}$ for which the function is differentiable on a neighbourhood (since C/R are necessary for differentiability), & hence f^n is nowhere analytic.

b) Here the f^n is given by $e^y e^{ix} = e^y \cos x + ie^y \sin x$,
So $u_x = -e^y \sin x$, $u_y = e^y \cos x$,
$v_x = e^y \cos x$, $v_y = e^y \sin$ (f^n is defined on $\mathbb{C}$)
Hence $C/R_I \Rightarrow -e^y \sin x = e^y \sin x \Leftrightarrow x = n\pi$, and the remainder of the argument proceeds as in part a).

3) a) For this f^n, we have $u = x^3$, $v = (1-y)^3$
$\Rightarrow u_x = 3x^2$, $u_y = 0$, $v_x = 0$, $v_y = -3(1-y)^2$.
So $C/R_I \Rightarrow 3x^2 = -3(1-y)^2$, which is only satisfied at $x=0, y=1$.
$C/R_{II} \Rightarrow 0 = 0$ holds everywhere. Note also that u & v and all the first-order partials exist everywhere.
Since C/R only hold at i & C/R are necessary for differentiability, we see the f^n is only possibly diff^{ble} at i. Hence in particular it is not diff^{ble} on any nbhd of any pt., & hence is nowhere analytic.

b) By inspection v, u, u_x, u_y, v_x, v_y are cts on $\mathbb{C}$. Since C/R hold at i, this means f^n is indeed diff^{ble} at i.

4. (a) $u(x,y) = \sqrt{|xy|}$, $v(x,y) = 0$.

$\Rightarrow v_x = v_y = 0$.

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = 0.$$

$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = 0.$$

So C/R hold at $(0,0)$.

(b) Consider $\Delta z = h(1+i)$.

Then $\dfrac{f(0+\Delta z)-f(0)}{\Delta z} = \dfrac{\sqrt{|h\cdot h|}}{h(1+i)}$

$= \dfrac{|h|}{h}\cdot\dfrac{1}{1+i}$ does not approach a limit as $h\to 0$ ($\to \frac{1}{1+i}$ as $h\to 0^+$, & $\to \frac{-1}{1+i}$ as $h\to 0^-$).

Hence $f'(0)$ can't exist.

(c) C/R is necessary but not sufficient for differentiability, so no contradiction.

⑥ a) Note $x_n, y_n \to 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{1}{1-(-\frac{1}{2})}$ (geom series) $= \frac{2}{3}$.

Hence: set $p = \frac{2}{3} + \frac{2}{3}i$, & assume $\Lambda \subset A \cup B$, where A, B open, disjoint, & $A \cap \Lambda \neq \emptyset$, $\Lambda \cap B \neq \emptyset$.

wlog $p \in B$.

Set $N = \min_{n \in \mathbb{N}} \{n : I_n \subset B\}$

Note that B is open, so $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(p) \subset B$, so $I_n \subset B$ $\forall n$ sufficiently large (since $x_n + iy_n \to p$). Hence N is well defined & finite, & indeed $N > 0$ (or else $\Lambda \subset B$).

Consider I_{N-1}, & set $q_t = t z_{N-1} + (1-t) z_N$, & define $\tau = \inf_{t \in [0,1]} \{t : q_s \in B \ \forall s \in [t,1]\}$

Note $q_1 = z_N \in B$ since $z_N \in I_N \subset B$, so τ is well defined.

$\tau = 1$ is not possible: since $z_N \in B \Rightarrow \exists$ some $\varepsilon' : B_\varepsilon(z_N) \subset B \Rightarrow \tau < 1$.

$I_{N-1} \not\subset B \Rightarrow \tau > 0$.

So $\tau \in (0,1)$. Now consider q_τ. If $q_\tau \in B$, we know $\exists$ some $B_{\varepsilon'}(q_\tau) \subset B$, giving a contradiction to the def^n of τ: if $q_\tau \in A$, we obtain a similar contradiction. Hence we can't find such A & B, & Λ is connected.

b) Any piecewise linear path in Λ connecting 0 to $\frac{2}{3} + \frac{2}{3}i$ must include an arbitrarily large number of the I_n's, so Λ is not affinely path connected in the sense given in class.

c) No contradiction. Piecewise affinely p.c. & ~~connected are equivalent for open sets~~, but Λ is not open (can check easily that e.g. $0+0i$ is a boundary point; Λ is in fact closed).