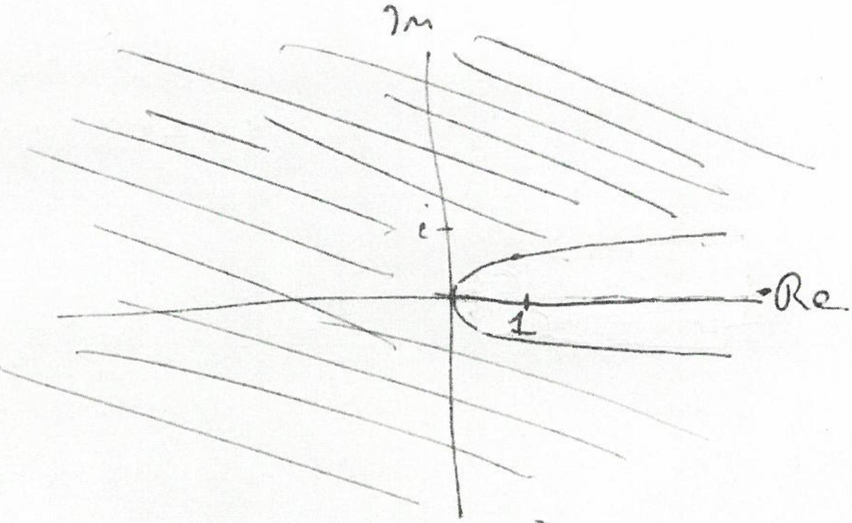
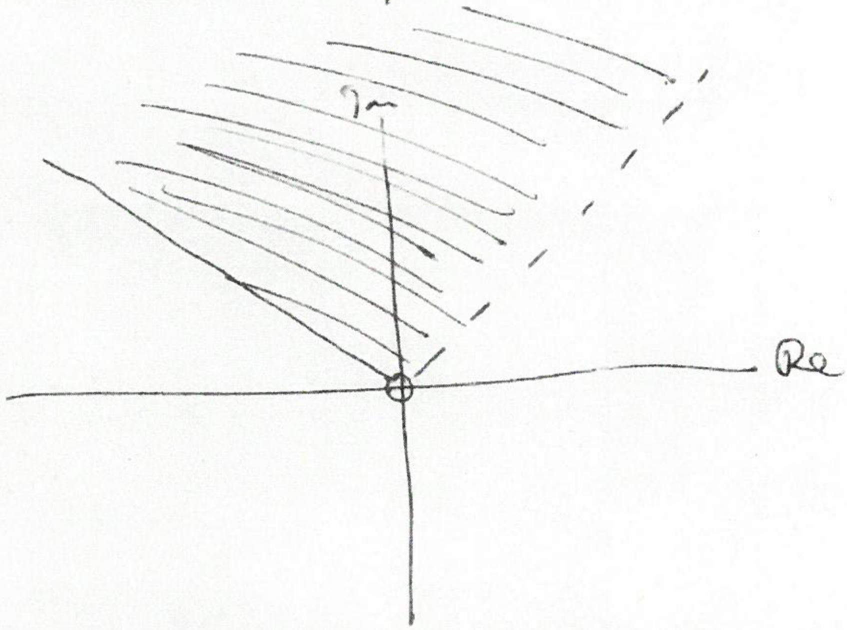
MATH3401 Asst 1 Sol^{n}s

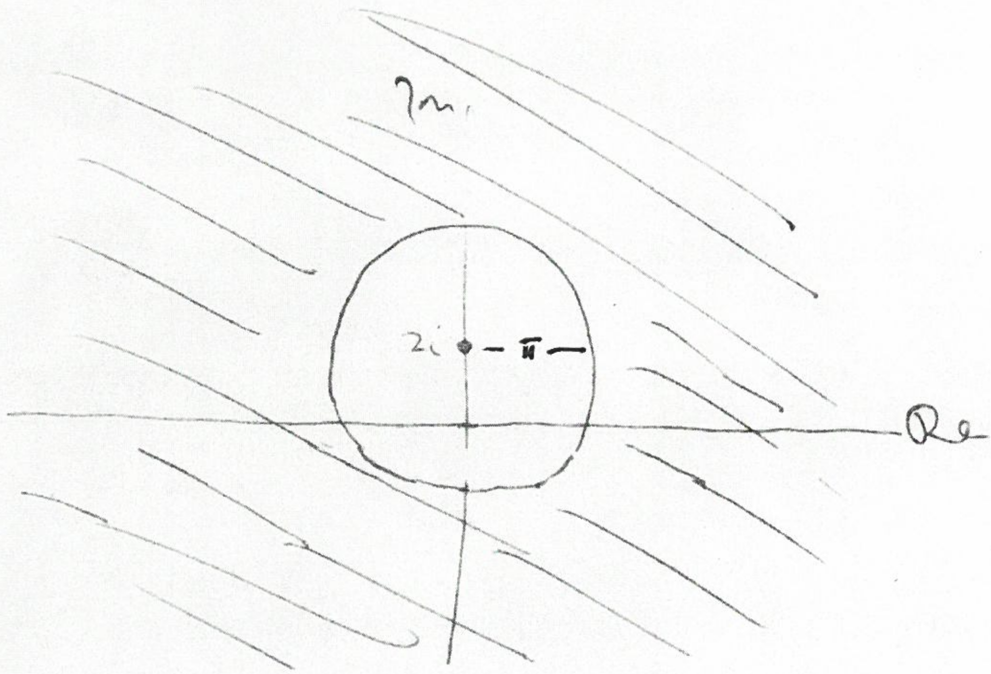
P1 a)



b)



c)



(P2) Put $z = x+iy$.

a) $\bar{z} = z \iff x+iy = x-iy \iff 2iy = 0 \iff y=0$
So, $\{z : \operatorname{Im}(z) = 0\}$.

b) $\bar{z} + z = 0 \iff 2x = 0 \iff x = 0$
So, $\{z : \operatorname{Re}(z) = 0\}$.

c) $\bar{z} = 4/z \iff z\bar{z} = 4 \iff |z|^2 = 4 \iff |z| = 2$.
So, $\{z : |z| = 2\}$.

(P3) a) $\dfrac{i}{(1-i)} + \dfrac{1-i}{i} = \dfrac{i^2+(1-i)^2}{i(1-i)} = \dfrac{-1-2i}{1+i} \cdot \dfrac{1-i}{1-i}$

$= \dfrac{-1+i-2i-2}{2} = \dfrac{-(3+i)}{2} = -\dfrac{3}{2} - i\dfrac{1}{2}$

b) $64i = 64^3 \exp(i\pi/2)$, so cube roots are

$\left\{ \exp\left[i\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right], k = 0,1,2 \right\}$

$= \left\{4\exp \frac{i\pi}{6}, 4\exp \frac{5\pi i}{6}, 4.\exp \frac{9\pi i}{6}\right\}$

$= \left\{4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}), 4(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}),\right.$
$\left. 4(\cos\frac{9\pi i}{6} + i\sin\frac{9\pi i}{6})\right\}$

$= \{2\sqrt{3} + 2i, -2\sqrt{3} + 2i, -4i\}$

c) $\left(\dfrac{i+1}{\sqrt{2}}\right)^{1337} = \left(\exp\dfrac{i\pi}{4}\right)^{1337} = \exp\dfrac{1337\pi i}{4}$

$= \exp\dfrac{\pi i}{4} = \dfrac{i+1}{\sqrt{2}}$

(P4) a) Given $p,q,r,s \in \mathbb{Q}$, WTS:

① $(p+q\sqrt{2})+(r+s\sqrt{2}) \in \mathbb{Q}(\sqrt{2})$ (closure under +);

② $(p+q\sqrt{2})(r+s\sqrt{2}) \in \mathbb{Q}(\sqrt{2})$ (closure under $\cdot$).

①: LHS $= (p+r)+(q+s)\sqrt{2}$, since $\mathbb{R}$ is a field, & $p+r \in \mathbb{Q}$, $q+s \in \mathbb{Q}$ since $\mathbb{Q}$ is a field. This shows ①.

②: LHS $= (pr+2sq)+(rq+ps)\sqrt{2}$, since $\mathbb{R}$ is a field, & $pr+2sq \in \mathbb{Q}$, $rq+ps \in \mathbb{Q}$ since $\mathbb{Q}$ is a field. This shows ②.

NEXT SHOW (F1)–(F7).

(F1) follows since $\mathbb{R}$ is a field.

(F2)(i) follows since $\mathbb{R}$ is a field, with $0 = 0+0\sqrt{\ } \in \mathbb{Q}(\sqrt{\ })$.

(F2)(ii) same: $-(p+q\sqrt{2}) = (-p)+(-q)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

(F4) follows since $\mathbb{R}$ is a field.

(F5)(i) same: $1 = 1+0\sqrt{2} \in \mathbb{Q}(\sqrt{2})$.

(ii) For $p,q \in \mathbb{Q}$ not both 0, we have

$$\frac{1}{p+q\sqrt{2}} = \frac{1}{p+q\sqrt{2}} \cdot \frac{p-q\sqrt{2}}{p-q\sqrt{2}} = \frac{p}{p^2-2q^2} + \left(\frac{-q}{p^2-2q^2}\right)\sqrt{2}$$

which is in $\mathbb{Q}(\sqrt{2})$; note in particular

$p^2 - 2q^2 \neq 0$ for $p, q \in \mathbb{Q}$, since $\sqrt{2} \notin \mathbb{Q}$.

(F6) & (F7) follow since $\mathbb{R}$ is a field.

Hence $\mathbb{Q}(\sqrt{2})$ is a field.

b) Suppose $\sqrt{3} = a + b\sqrt{2}$, $\circledast$ $a, b \in \mathbb{Q}$.

Note $b \neq 0$. (If $b = 0$, then $\sqrt{3} = a \in \mathbb{Q}$, but $\sqrt{3} \notin \mathbb{Q}$).

Hence, we can rewrite $\circledast$ as

$\sqrt{3} - b\sqrt{2} = a$, & on squaring,

$2b^2 + 3 - 2\sqrt{2}\sqrt{3}\, b = a^2$

i.e., $2\sqrt{6}\, b = 3 - a^2 + 2b^2$

since $b \neq 0$, this yields $\sqrt{6} = \dfrac{3 - a^2 + 2b^2}{2b}$, i.e.,

$\sqrt{6} \in \mathbb{Q}$, C!

(P5) As per hint: assume $\mathbb{C}$ is well ordered. note $(-1) \notin P$, because if it were, then $(-1)(-1) \in P$ (P closed under multiplication), i.e., $1 \in P$, so both $1 \in P$ & $-1 \in P$, which is impossible.

So if $i \in P$, then $(i)(i) = -1 \in P$ (again, by closure of P under $\times$), which it isn't.
But if $-i \in P$, then $(-i)(-i) = -1 \in P$, which it isn't.
Since $i \neq 0$, we thus have a contradiction.

$\square$