SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 9

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(1) Are the following functions conformal? To answer this, analyse their domains and draw some sketches to map specific regions.

(a) $f(z) = e^z$

(b) $f(z) = z^2$

(c) $f(z) = z + \frac{1}{z}$

(2) Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate when

(a) $u(x, y) = 2x(1 - y)$

(b) $u(x, y) = 2x - x^3 + 3xy^2$

(c) $u(x, y) = \sinh x \sin y$

(c) $u(x, y) = \frac{x}{x^2 + y^2}$

(3) Let $f(z)$ be an analytic function on a domain Ω that does not include the origin. Using polar coordinates in Ω, f has the form

$$f(z) = u(r, \theta) + iv(r, \theta).$$

(a) Using the chain rule, show that all partial derivatives of u and v of first and second order with respect to r and/or θ are continuous (indeed, all partial derivatives of any order are).

(b) Using the Cauchy-Riemann equations in polar coordinates, show that u satisfies

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

This is the polar form of *Laplace's equation*, after having multiplied through by r^2: the *Laplacian* Δ is given in spherical coordinates by $\frac{1}{r^2}(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2})$.

(c) Show that v satisfies

$$r^2 v_{rr} + r v_r + v_{\theta\theta} = 0.$$

(d) Give a procedure which finds the harmonic conjugate of a given harmonic function u given in polar coordinates (don't transform to cartesian coordinates: the harmonic conjugate v should be expressed as $v(r, \theta)$).

(e) Verify directly that the function $u(r, \theta) = \ln(r^2)$ is harmonic on the domain $\{z \mid r > 0, 0 < \arg z < 2\pi\}$, and use your procedure from part (d) to calculate a harmonic conjugate.