SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 9

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(1) Are the following functions conformal? To answer this, analyse their domains and draw some sketches to map specific regions.

(a) $f(z) = e^z$

(b) $f(z) = z^2$

(c) $f(z) = z + \dfrac{1}{z}$

Solution. (a) The function $f(z) = e^z$ is conformal throughout the entire z plane since the function is entire and $(e^z)' = e^z \neq 0$ for each z. For details about this mapping see Section 14 from Churchill's book.

(b) We know that the function $f(z) = z^2$ maps a quarter plane to a half plane, and therefore doubles the angle between the coordinate axes at the origin (see Figures 1 and 2).

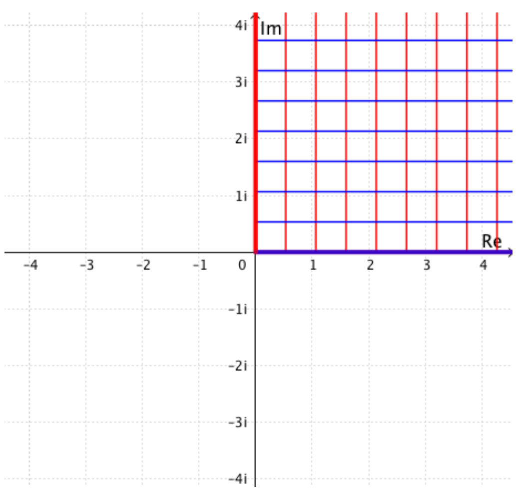

Figure 1: A quarter plane.

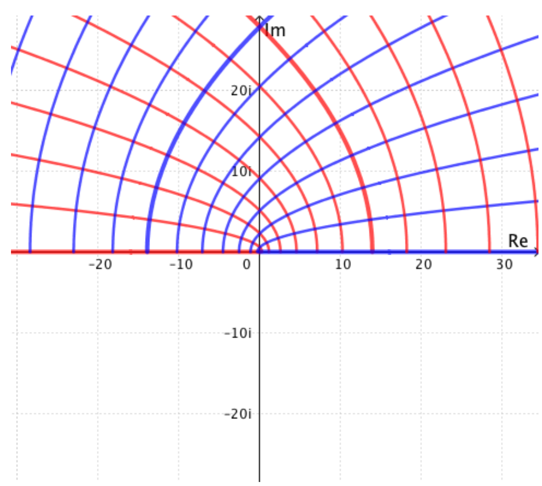

Figure 2: Image under the map $f(z) = z^2$.

The function $f(z) = z^2$ is conformal on $\mathbb{C}$ except at the origin, since f is entire, and 0 is the only critical point of f.

Note that, due to conformality, the map preserves angles everywhere else.

Although $f(z) = z^2$ is not conformal at $z_0 = 0$, we can find a region that will be mapped conformally. For example, consider the right half-plane $\{\text{Re }(z) > 0\}$. This region is mapped conformally by $w = z^2$ onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$, as illustrated in Figures 3 and 4.

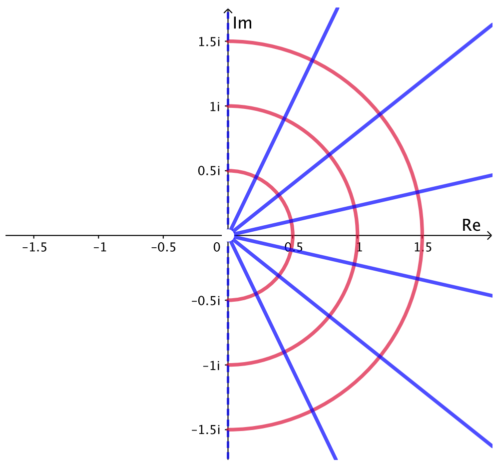

Figure 3: $\{\text{Re }(z) > 0\}$

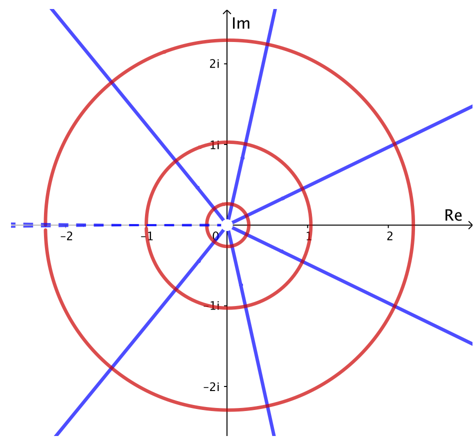

Figure 4: Image under the map $f(z) = z^2$.

(c) Consider now the **Joukowsky map**

$$w = z + \frac{1}{z} \tag{1}$$

Since

$$\frac{d}{dz}w = 1 - \frac{1}{z^2} = 0 \quad \text{if and only if} \quad z = \pm 1,$$

the **Joukowsky map** is conformal except at the critical points $z = \pm 1$ as well as the singularity $z = 0$, where it is not defined.

If $z = e^{i\theta}$ lies on the unit circle, then

$$w = e^{i\theta} + e^{-i\theta} = 2\cos\theta,$$

lies on the real axis, with $-2 \leq w \leq 2$. Thus, the **Joukowsky map** squashes the unit circle down to the real line segment $[-2, 2]$. The images of points outside the unit circle fill the rest of the w plane, as do the images of the (nonzero) points inside the unit circle. Indeed, if we solve (1) for z, we have

$$z = \frac{1}{2}\left(w \pm \sqrt{w^2 - 4}\right).$$

We see that every w except ± 2 comes from two different points z; for w not on the critical line segment $[-2, 2]$, one point (with the minus sign) lies inside and one (with the plus sign) lies outside the unit circle, whereas if $-2 < w < 2$, both points lie on the unit circle and a common vertical line.

Therefore, the **Joukowski map**

$$f(z) = z + \frac{1}{z}$$

defines a one-to-one conformal mapping from the exterior of the unit circle $\{|z| > 1\}$ onto the exterior of the line segment $\mathbb{C} \setminus [-2, 2]$.

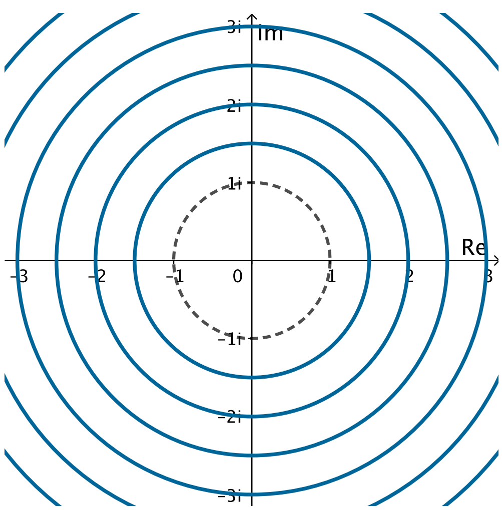

Figure 5: Concentric circles $|z| = r \geq 1$.

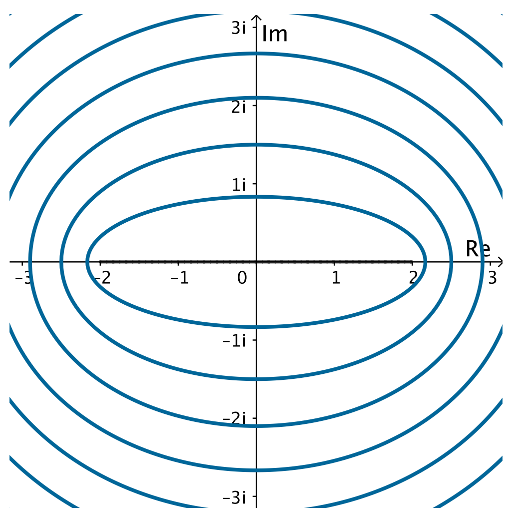

Figure 6: Image under the **Joukowski map**.

(2) Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate when

(a) $u(x,y) = 2x(1-y)$

(b) $u(x,y) = 2x - x^3 + 3xy^2$

(c) $u(x,y) = \sinh x \sin y$

(d) $u(x,y) = \dfrac{x}{x^2+y^2}$

Solutions:

(a) When $u(x,y) = 2x(1-y)$, we have that

$$u_x = 2 - 2y, \qquad u_y = -2x$$

and

$$u_{xx} = 0, \qquad u_{yy} = 0.$$

Thus

$$u_{xx} + u_{yy} = 0.$$

To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2-2y$. Now, using Cauchy-Riemann equations

$$u_x = v_y \implies v_y = 2 - 2y \implies v(x,y) = 2y - y^2 + g(x).$$

Then

$$u_y = -v_x \implies -2x = -g'(x) \implies g'(x) = 2x \implies g(x) = x^2 + c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x,y) = 2y - y^2 + (x^2 + c) = x^2 - y^2 + 2y + c \quad (c \in \mathbb{R}).$$

(b) When $u(x,y) = 2x - x^3 + 3xy^2$, we have that

$$u_x = 2 - 3x^2 + 3y^2, \qquad u_y = 6xy$$

and

$$u_{xx} = -6x, \qquad u_{yy} = 6x.$$

Thus $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \implies v_y = 2 - 3x^2 + 3y^2 \implies v(x,y) = 2y - 3x^2y + y^3 + g(x).$$

Then

$$u_y = -v_x \implies 6xy = 6xy - g'(x) \implies g'(x) = 0 \implies g(x) = c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x,y) = 2y - 3x^2y + y^3 + c \quad (c \in \mathbb{R}).$$

(c) When $u(x,y) = \sinh x \sin y$, we have that

$$u_x = \cosh x \sin y, \qquad u_y = \sinh x \cos y$$

and

$$u_{xx} = \sinh x \sin y, \qquad u_{yy} = -\sinh x \sin y.$$

Thus $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = \cosh x \sin y$. Now

$$u_x = v_y \implies v_y = \cosh x \sin y \implies v(x,y) = -\cosh x \cos y + g(x).$$

Then

$$u_y = -v_x \implies \sinh x \cos y = \sinh x \cos y - g'(x) \implies g'(x) = 0 \implies g(x) = c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x,y) = -\cosh x \cos y + c \quad (c \in \mathbb{R}).$$

(d) Finally for $u(x,y) = \dfrac{x}{x^2+y^2}$, we have that

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2}, \qquad u_y = \frac{-2xy}{(x^2+y^2)^2}$$

and

$$u_{xx} = 2x\frac{x^2-3y^2}{(x^2+y^2)^3}, \qquad u_{yy} = -2x\frac{x^2-3y^2}{(x^2+y^2)^3}.$$

Thus $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. Now

$$u_x = v_y \implies v_y = \frac{y^2-x^2}{(x^2+y^2)^2} \implies v(x,y) = -\frac{y}{x^2+y^2} + g(x).$$

Then

$$u_y = -v_x \implies \frac{-2xy}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2} - g'(x)$$
$$\implies g'(x) = 0 \implies g(x) = c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x,y) = -\frac{y}{x^2+y^2} + c \quad (c \in \mathbb{R}).$$

(3) Let $f(z)$ be an analytic function on a domain Ω that does not include the origin. Using polar coordinates in Ω, f has the form

$$f(z) = u(r, \theta) + iv(r, \theta).$$

(a) Using the chain rule, show that all partial derivatives of u and v of first and second order with respect to r and/or θ are continuous (indeed, all partial derivatives of any order are).

(b) Using the Cauchy-Riemann equations in polar coordinates, show that u satisfies

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

This is the polar form of *Laplace's equation*, after having multiplied through by r^2: the *Laplacian* Δ is given in spherical coordinates by $\frac{1}{r^2}(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2})$.

(c) Show that v satisfies

$$r^2 v_{rr} + r v_r + v_{\theta\theta} = 0.$$

(d) Give a procedure which finds the harmonic conjugate of a given harmonic function u given in polar coordinates (don't transform to cartesian coordinates: the harmonic conjugate v should be expressed as $v(r, \theta)$).

(e) Verify directly that the function $u(r, \theta) = \ln(r^2)$ is harmonic on the domain $\{z \mid r > 0, 0 < \arg z < 2\pi\}$, and use your procedure from part (d) to calculate a harmonic conjugate.

Solution: (a) Since $f(z)$ is analytic on Ω, $f(z)$ is also differentiable on Ω. Then the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighbourhood of a given nonzero point $z_0 \in \Omega$ and are continuous at z_0.

Now the chain rule for differentiating real-valued functions of two real variables can be used to write polar form of $u(r, \theta)$ and $v(r, \theta)$ in terms of the ones with respect to x and y.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

Thus

$$u_r = u_x \cos\theta + u_y \sin\theta$$

and

$$u_\theta = -u_x\, r\sin\theta + u_y\, r\cos\theta.$$

This means that u_r and u_θ are continuous.

Now, in a similar way, using the following expressions

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

we obtain

$$v_r = v_x \cos\theta + v_y \sin\theta$$

and

$$v_\theta = -v_x\, r\sin\theta + v_y\, r\cos\theta.$$

Hence, v_r and v_θ are also continuous.

(b) The Cauchy-Riemann equations in polar coordinates are

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

Then

$$ru_r = v_\theta \implies ru_{rr} + u_r = v_{\theta r} \implies r^2u_{rr} + ru_r = rv_{\theta r}$$

and

$$u_\theta = -rv_r \implies u_{\theta\theta} = -rv_{r\theta}.$$

Adding these two equations we obtain

$$r^2u_{rr} + ru_r + u_{\theta\theta} = rv_{\theta r} - rv_{r\theta}.$$

Now, since $v_{r\theta} = v_{\theta r}$, we have

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0.$$

(c) This can be shown in a similar way to part (b).

(d) We can use Cauchy-Riemann equations in polar coordinates

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

(e) Here we can use part (b). If $u(r,\theta) = \ln(r^2) = 2\ln r$, with $r > 0$, then

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = r^2\left(-\frac{2}{r^2}\right) + r\left(\frac{2}{r}\right) + 0 = 0$$

This means that the function $u(r,\theta) = \ln(r^2)$ is harmonic on the domain $\{z \,|\, r > 0, 0 < \arg z < 2\pi\}$.

Now, from Cauchy-Riemann equation $ru_r = v_\theta$ and the derivative $u_r = \frac{2}{r}$, we obtain $v_\theta = 2$. Then

$$v(r,\theta) = 2\theta + g(r)$$

where $g(r)$ is an arbitrary differentiable function of r. Using the other Cauchy-Riemann equation $u_\theta = -rv_r$ we get $0 = -rg'(r)$. In other words, $g'(r) = 0$ and so $g(r) = c$, with $c \in \mathbb{R}$. Therefore $v(r,\theta) = 2\theta + c$ is a harmonic conjugate of $u(r,\theta) = \ln(r^2)$.