SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 8

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(1) Evaluate $\displaystyle\int_C f(z)\,dz$ for the following functions f and contours C.

(a) $f(z) = \pi \exp(\pi \overline{z})$ and C is the boundary of the square with vertices at the points

$$0,\ 1,\ 1+i,\ \text{ and }\ i.$$

The orientation of C being in the counterclockwise direction.

(b) $f(z)$ is the branch

$$z^{-1+i} = \exp\left[(-1+i)\,\log z\right] \quad (|z| > 0,\ 0 < \arg z < 2\pi)$$

of the indicated power function, and C is unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

(c) $f(z)$ is the principal branch

$$z^{i} = \exp\left[i \,\mathrm{Log}\, z\right] \quad (|z| > 0,\ -\pi < \mathrm{Arg}\, z < \pi)$$

of this power function, and C is semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

(2) Evaluate the integral $\int_C \mathrm{Re}(z)\,dz$ for the following contours C from -4 to 4:

1. The line segments from -4 to $-4-4i$ to $4-4i$ to 4;
2. the lower half of the circle with radius 4, centre 0;
3. the upper half of the circle with radius 4, centre 0.
4. What conclusions (if any) can you draw about the function $z \mapsto \mathrm{Re}(z)$ from this?

(3) Evaluate the following integrals, justifying your procedures. For b) you should also state why the integral is well defined (i.e., independent of the path taken).

(a) $\displaystyle\int_C \left(e^z + \frac{1}{z}\right) dz$, where C is the lower half of the circle with radius 1, centre 0, negatively oriented;

(b) $\displaystyle\int_{\pi i}^{2\pi i} \cosh z dz$.

(4) Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz\right| \leq \frac{\pi R\left(2R^2 + 1\right)}{\left(R^2 - 1\right)\left(R^2 - 4\right)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

(5) Show that

$$\int_{-1}^{1} z^i\, dz = \frac{1 + e^{-\pi}}{2}(1 - i)$$

where the integrand denotes the principal branch

$$z^i = \exp\left[i \operatorname{Log} z\right] \quad \left(|z| > 0,\ -\pi < \operatorname{Arg} z < \pi\right)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with problem 1c).

Suggestion: Try to use an antiderivative of the branch

$$z^i = \exp\left[i \log z\right] \quad \left(|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right).$$

(6) Find the value of the integral of $f(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $f(z) = \dfrac{1}{z^2 + 4}$;

(b) $f(z) = \dfrac{1}{(z^2 + 4)^2}$.

Suggestion: Use Cauchy Integral Formula and its extension.