SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 8

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(1) Evaluate $\int_C f(z)\,dz$ for the following functions f and contours C.

(a) $f(z) = \pi \exp(\pi \overline{z})$ and C is the boundary of the square with vertices at the points

$$0,\ 1,\ 1+i,\ \text{ and }\ i.$$

The orientation of C being in the counterclockwise direction.

(b) $f(z)$ is the branch

$$z^{-1+i} = \exp\left[(-1+i)\ \log z\right] \quad (|z| > 0,\, 0 < \arg\, z < 2\pi)$$

of the indicated power function, and C is unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

(c) $f(z)$ is the principal branch

$$z^i = \exp\left[i\,\text{Log}\ z\right] \quad (|z| > 0,\, -\pi < \text{Arg}\ z < \pi)$$

of this power function, and C is semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Solution: (a) In this problem, the path C is the sum of the paths C_1, C_2, C_3, and C_4 that are shown in Figure 1.

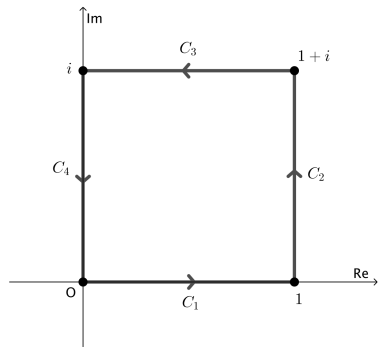

Figure 1: C is the boundary of the square with vertices at the points $0,\ 1,\ 1+i,$ and i.

The functions to be integrated around the closed path C is $f(z) = \pi e^{\pi \overline{z}}$. Notice that $C = C_1 + C_2 + C_3 + C_4$. Thus we need to find the values of the integrals along the individual legs of the square C.

(i) Since C_1 is $z = x$, with $0 \leq x \leq 1$,

$$\int_{C_1} \pi e^{\pi \overline{z}}\, dz = \pi \int_0^1 e^{\pi x}\, dx = e^{\pi} - 1.$$

(ii) Since C_2 is $z = 1 + iy$, with $0 \leq y \leq 1$,

$$\int_{C_2} \pi e^{\pi \overline{z}}\, dz = \pi \int_0^1 e^{\pi(1-iy)} i\, dy = e^{\pi} \pi i \int_0^1 e^{-i\pi y}\, dy = 2e^{\pi}.$$

(iii) Since C_3 is $z = (1-x) + i$, with $0 \leq x \leq 1$,

$$\int_{C_3} \pi e^{\pi \overline{z}}\, dz = \pi \int_0^1 e^{\pi[(1-x)-i]}(-1)\, dx = \pi e^{\pi} \int_0^1 e^{-\pi x}\, dx = e^{\pi} - 1.$$

(iv) Since C_4 is $z = i(1-y)$, with $0 \leq y \leq 1$,

$$\int_{C_4} \pi e^{\pi \overline{z}}\, dz = \pi \int_0^1 e^{-\pi(1-y)i}(-i)\, dx = \pi i \int_0^1 e^{i\pi y} = -2.$$

Finally, since

$$\int_C \pi e^{\pi \overline{z}}\, dz = \int_{C_1} \pi e^{\pi \overline{z}}\, dz + \int_{C_2} \pi e^{\pi \overline{z}}\, dz + \int_{C_3} \pi e^{\pi \overline{z}}\, dz + \int_{C_4} \pi e^{\pi \overline{z}}\, dz,$$

we find that

$$\int_C \pi e^{\pi \overline{z}}\, dz = 4(e^{\pi} - 1).$$

(b) To integrate the branch

$$z^{-1+i} = \exp\left[(-1+i)\,\log z\right] \quad (|z| > 0,\ 0 < \arg z < 2\pi)$$

around the circle $C : z = e^{i\theta}$ with $0 \le \theta \le 2\pi$, write

$$\begin{aligned} \int_C z^{-1+i}\, dz &= \int_C e^{(-1+i)\log z}\, dz = \int_0^{2\pi} e^{(-1+i)(\ln 1 + i\theta)} i e^{i\theta}\, d\theta \\ &= i \int_0^{2\pi} e^{-i\theta - \theta} e^{i\theta}\, d\theta = i \int_0^{2\pi} e^{-\theta}\, d\theta \\ &= i\left(1 - e^{-2\pi}\right). \end{aligned}$$

Remark 1: The integral $\int_C z^{-1+i} dz$ exists even though the branch z^{-1+i} with $|z| > 0$ and $0 < \arg z < 2\pi$ is not defined at the point $z = 1$ of the contour. Why?

(c) To integrate the branch

$$z^i = \exp\left[i \,\text{Log}\, z\right] \quad (|z| > 0, \, -\pi < \text{Arg}\; z < \pi)$$

around the semicircle $C : z = e^{i\theta}$ with $0 \le \theta \le \pi$, write

$$\begin{aligned}
\int_C z^i\, dz &= \int_C e^{i\,\text{Log}\; z}\, dz = \int_0^\pi e^{i(\ln 1 + i\theta)} i e^{i\theta}\, d\theta \\
&= i \int_0^\pi e^{i\theta - \theta}\, d\theta = i \int_0^\pi e^{(i-1)\theta}\, d\theta \\
&= \frac{i}{i-1}\left[e^{(i-1)\pi} - 1\right] = \frac{i}{i-1}\left[-e^{-\pi} - 1\right] \\
&= \frac{i}{1-i}(1 + e^{-\pi}) \\
&= -\frac{1 + e^{-\pi}}{2}\,(1 - i).
\end{aligned}$$

Note: The last expression was obtained to compare with problem 5.

Remark 2: The integral $\int_C z^i dx$ exists even though the branch z^i with $|z| > 0$ and $-\pi < \text{Arg}\; z < \pi$ is not defined at the end point $z = -1$ of the contour. Why?

(2) Evaluate the integral $\int_C \text{Re}(z)\, dz$ for the following contours C from -4 to 4:

(a) The line segments from -4 to $-4-4i$ to $4-4i$ to 4;

(b) the lower half of the circle with radius 4, centre 0;

(c) the upper half of the circle with radius 4, centre 0.

(d) What conclusions (if any) can you draw about the function $z \mapsto \text{Re}(z)$ from this?

Solution: (a) Notice that the contour C consists of three contours:

i. C_1 defined by $z(t) = -4 - 4it$, with $0 \le t \le 1$, followed by

ii. C_2 defined by $z(t) = -4(1-2t) - 4i$, with $0 \le t \le 1$, and finally

iii. C_3 defined by $z(t) = 4 - 4i(1-t)$, with $0 \le t \le 1$.

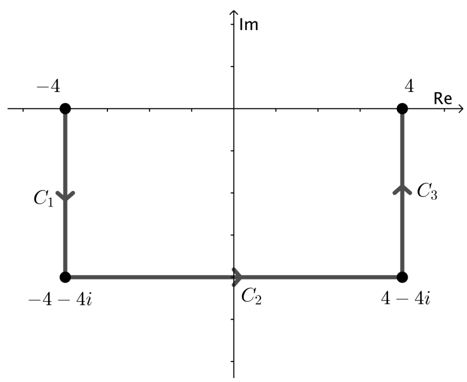

Figure 2: C is line segments from -4 to $-4-4i$ to $4-4i$ to 4.

Thus

$$\int_C f(z)dz = \int_{C_1+C_2+C_3} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz \tag{1}$$

where $f(z) = \text{Re}(z)$. Recall also that $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$.

Now, for C_1 we have that $z'(t) = -4i$. Then

$$\int_{C_1} f(z)dz = \int_0^1 (-4)(-4i)dt = 16i \int_0^1 dt = 16it\Big|_0^1 = 16i.$$

For C_2 we have that $z'(t) = 8$, then

$$\begin{aligned}
\int_{C_2} f(z)dz &= \int_0^1 [-4(1-2t)]\,(8)dt \\
&= \int_0^1 (64t - 32)dt \\
&= 64\int_0^1 tdt - 32\int_0^1 dt \\
&= 64\frac{t^2}{2}\Big|_0^1 - 32t\Big|_0^1 \\
&= 32 - 32 = 0
\end{aligned}$$

Finally for C_3 we have that $z'(t) = 4i$, then

$$\int_{C_3} f(z)dz = \int_0^1 (4)(4i)dt = 16i\int_0^1 dt = 16it\Big|_0^1 = 16i.$$

Therefore, using expression 1, we obtain

$$\int_C \mathrm{Re}(z)\,dz = 32i.$$

(b) In this case, the contour C is defined by

$$z(t) = 4e^{it} = 4\cos t + 4i\sin t,$$

with $\pi \leq t \leq 2\pi$. Here we have $z'(t) = 4ie^{it}$.

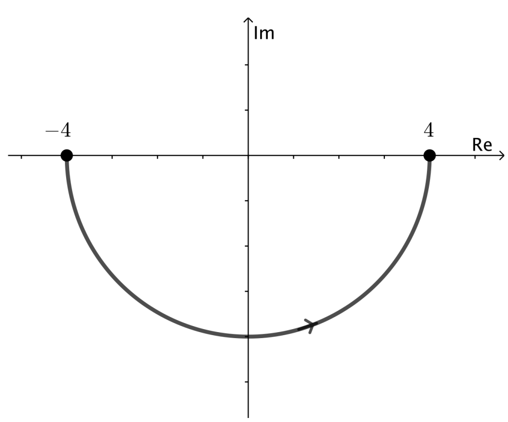

Figure 3: C is the lower half of the circle with radius 4, centre 0.

Thus

$$
\begin{aligned}
\int_C \mathrm{Re}(z)\, dz &= \int_{\pi}^{2\pi} (4\cos t)(4ie^{it})dt \\
&= 16i \int_{\pi}^{2\pi} \cos t e^{it} dt \\
&= 16i \int_{\pi}^{2\pi} \cos^2 t dt - 16 \int_{\pi}^{2\pi} \cos t \sin t dt \\
&= 8i \int_{\pi}^{2\pi} [1 + \cos(2t)]dt - 16 \cdot \left.\frac{\sin^2 t}{2}\right|_{\pi}^{2\pi} \\
&= 8i \left[t + \frac{\sin(2t)}{2} \right] \Bigg|_{\pi}^{2\pi} - 0 = 8\pi\, i
\end{aligned}
$$

(c) Finally, in this case, the contour C is defined by

$$z(t) = -4e^{-it} = -4\cos t + 4i \sin t,$$

with $0 \leq t \leq \pi$. Here we have $z'(t) = 4ie^{-it}$.

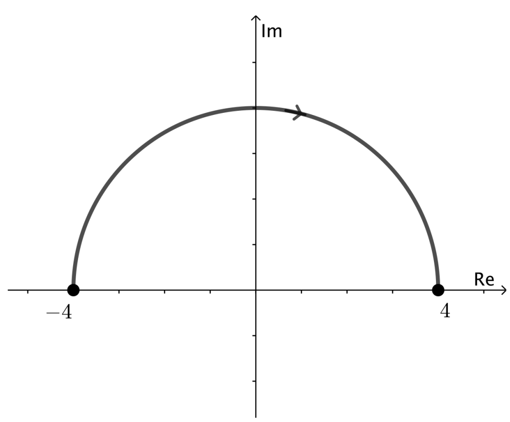

Figure 4: C is the upper half of the circle with radius 4, centre 0.

Thus

$$\begin{aligned}
\int_C \mathrm{Re}(z)\,dz &= \int_0^{\pi} (-4\cos t)(4ie^{-it})dt \\
&= -16i\int_0^{\pi} \cos t e^{-it} dt \\
&= -16i\int_0^{\pi} \cos^2 t dt + 16\int_0^{\pi} \cos t \sin t dt \\
&= -8i\int_0^{\pi} [1+\cos(2t)]dt + 16\cdot \frac{\sin^2 t}{2}\bigg|_0^{\pi} \\
&= -8i\left[t+\frac{\sin(2t)}{2}\right]\bigg|_0^{\pi} + 0 = -8\pi\, i
\end{aligned}$$

(d) We have seen that the integral along each contour has a different value, thus we can conclude that $z \mapsto \mathrm{Re}(z)$ is not analytic on any domain containing 2 or more of the curves from (a), (b) and (c). (In fact, it is not analytic anywhere).

(3) Evaluate the following integrals, justifying your procedures. For (b) you should also state why the integral is well defined (i.e., independent of the path taken).

(a) $\displaystyle\int_C \left(e^z + \frac{1}{z}\right) dz$, where C is the lower half of the circle with radius 1, centre 0, negatively oriented;

(b) $\displaystyle\int_{\pi i}^{2\pi i} \cosh z \, dz$.

Solution: (a) Notice that the integrand $f(z) = e^z + 1/z$ is analytic on C. The function

$$F(z) = e^z + \log z$$

serves as an antiderivative of $f(z)$. Here $\log z$ is a branch of the logarithm chosen with the branch cut on the positive imaginary axis. That is,

$$\log z = \ln r + i\theta, \qquad \left(r > 0, \; \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right).$$

Thus

$$\int_C \left(e^z + \frac{1}{z}\right) dz = (e^z + \log z)\Big|_1^{-1} = \frac{1}{e} + \pi i - e - 2\pi i = \frac{1}{e} - e - \pi i.$$

(b) Since the integrand $f(z) = \cosh z$ is analytic, the integral is path independent. An antiderivative of $f(z)$ is

$$F(z) = \sinh z.$$

Thus

$$\int_C \cosh z \, dz = \sinh z \Big|_{\pi i}^{2\pi i} = \sinh(2\pi i) - \sinh(\pi i) = 0.$$

(4) Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{\pi R \left(2R^2 + 1\right)}{\left(R^2 - 1\right)\left(R^2 - 4\right)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

Solution: Note that if $|z| = R$ $(R > 2)$, then

$$|2z^2 - 1| \leq 2|z|^2 + 1 = 2R^2 + 1$$

and

$$|z^4 + 5z^2 + 4| = |z^2 + 1| \cdot |z^2 + 4| \geq \big||z|^2 - 1\big| \cdot \big||z|^2 - 4\big| = \left(R^2 - 1\right)\left(R^2 - 4\right).$$

Thus

$$\left| \frac{2z^2 - 1}{z^4 + 5z^2 + 4} \right| = \frac{|2z^2 - 1|}{|z^4 + 5z^2 + 4|} \leq \frac{2R^2 + 1}{\left(R^2 - 1\right)\left(R^2 - 4\right)}$$

when $|z| = R$ $(R > 2)$. Since the length of C_R is πR, then

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{\pi R \left(2R^2 + 1\right)}{\left(R^2 - 1\right)\left(R^2 - 4\right)} = \frac{\frac{\pi}{R}\left(2 + \frac{1}{R^2}\right)}{\left(1 - \frac{1}{R^2}\right)\left(1 - \frac{4}{R^2}\right)}$$

Hence we can conclude that the value of the integral tends to zero as R tends to infinity.

(5) Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i)$$

where the integrand denotes the principal branch

$$z^i = \exp\left[i \operatorname{Log} z\right] \quad (|z| > 0, \, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with problem 1c).

Suggestion: Try to use an antiderivative of the branch

$$z^i = \exp\left[i \log z\right] \quad \left(|z| > 0, \, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right).$$

Solution: Let C denote any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. This problem asks us to evaluate the integral

$$\int_{-1}^{1} z^i \, dz,$$

where z^i denotes the principal branch

$$z^i = \exp\left[i \operatorname{Log} z\right] \quad (|z| > 0, \, -\pi < \operatorname{Arg} z < \pi).$$

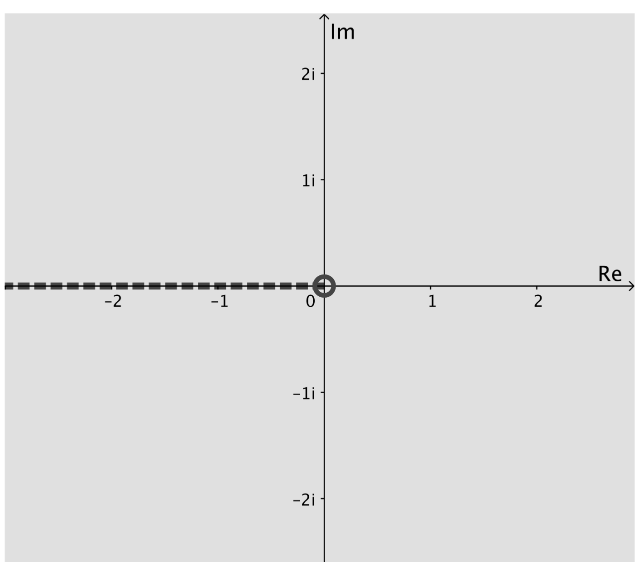

Figure 5: Principal branch of z^i.

An antiderivative of this branch *cannot* be used since the branch is not even defined at $z = -1$. But the integrand can be replaced by the branch

$$z^i = \exp[i \log z] \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

since it agrees with the integrand along C.

Using an antiderivative of this new branch, we can now write

$$\begin{aligned}
\int_{-1}^{1} z^i dz &= \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} = \frac{1}{i+1}\left[(1)^{i+1} - (-1)^{i+1}\right] \\
&= \frac{1}{i+1}\left[e^{(i+1)\log 1} - e^{(i+1)\log(-1)}\right] \\
&= \frac{1}{i+1}\left[e^{(i+1)(\ln 1 + i0)} - e^{(i+1)(\ln 1 + i\pi)}\right] \\
&= \frac{1}{i+1}\left(1 - e^{-\pi}e^{i\pi}\right) = \frac{1+e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i} \\
&= \frac{1+e^{-\pi}}{2}(1-i).
\end{aligned}$$

Note: Compare with problem 1c.

(6) Find the value of the integral of $f(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $f(z) = \dfrac{1}{z^2 + 4}$;

(b) $f(z) = \dfrac{1}{(z^2 + 4)^2}$.

Suggestion: Use Cauchy Integral Formula and its extension.

Solution: Let C denote the positively oriented circle $|z - i| = 2$, shown in Figure 6.

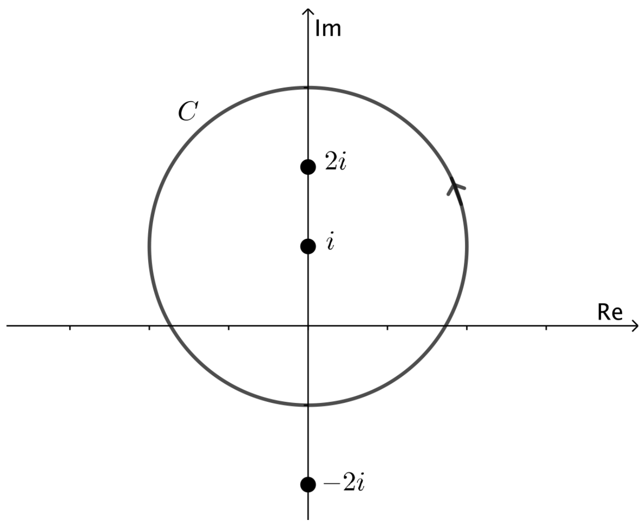

Figure 6: $|z - i| = 2$

(a) Using the Cauchy Integral Formula, we can write

$$\begin{aligned}
\int_C \frac{dz}{z^2+4}\,dz &= \int_C \frac{dz}{(z-2i)(z+2i)}\,dz = \int_C \frac{1/(z+2i)}{z-2i}\,dz \\
&= 2\pi i \left(\frac{1}{z+2i}\right)_{z=2i} \\
&= 2\pi i \left(\frac{1}{4i}\right) = \frac{\pi}{2}.
\end{aligned}$$

(b) Applying the extended form of the Cauchy Integral Formula, we have

$$
\begin{aligned}
\int_C \frac{dz}{(z^2+4)^2}\,dz &= \int_C \frac{dz}{(z-2i)^2(z+2i)^2}\,dz = \int_C \frac{1/(z+2i)^2}{(z-2i)^{1+1}}\,dz \\
&= \frac{2\pi i}{1!}\left[\frac{d}{dz}\frac{1}{(z+2i)^2}\right]_{z=2i} \\
&= 2\pi i\left[\frac{-2}{(z+2i)^3}\right]_{z=2i} \\
&= \frac{-4\pi i}{(4i)^3} = \frac{-4\pi i}{-(16)(4)i} = \frac{\pi}{16}.
\end{aligned}
$$