SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 7

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(1) Evaluate the following integrals:

a) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

b) $\displaystyle\int_0^{\pi/6} e^{i2t} dt$

Solution.

$$\begin{aligned}\int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i\int_1^2 \frac{1}{t} dt \\ &= -\frac{1}{2} - 2i\ln(2) = -\frac{1}{2} - i\ln(4)\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/6} e^{i2t} dt &= \left[\frac{e^{i2t}}{2i}\right]_0^{\pi/6} \\ &= \frac{1}{2i}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1\right) \\ &= \frac{\sqrt{3}}{4} + \frac{i}{4}\end{aligned}$$

(2) Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$$

Solution. First, notice that

$$I = \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta.$$

If $m = n$, I becomes

$$I = \int_0^{2\pi} d\theta = 2\pi.$$

If $m \neq n$, then

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)} \right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.$$

(3) Evaluate $\int_C f(z)dz$ for $f(z) = (z+2)/z$ and C is

a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Solution. For part a) we have

$$\begin{aligned}
\int_C \frac{z+2}{z} dz &= \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^\pi \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta} d\theta \\
&= 2i \int_0^\pi \left(e^{i\theta} + 1\right) d\theta \\
&= 2i \left[\frac{e^{i\theta}}{i} + \theta\right]_0^\pi = 2i(i + \pi + i) \\
&= -4 + 2\pi i
\end{aligned}$$

For part b) we have

$$\begin{aligned}
\int_C \frac{z+2}{z} dz &= 2i \int_\pi^{2\pi} \left(e^{i\theta} + 1\right) d\theta \\
&= \left[\frac{e^{i\theta}}{i} + \theta\right]_\pi^{2\pi} = 2i(-i + 2\pi - i - \pi) \\
&= 4 + 2\pi i
\end{aligned}$$

For part c) we just add the previous results to obtain $4\pi i$.

(4) Find the contour integral $\int_C \overline{z} dz$ for

(a) C is the triangle XYZ oriented counterclockwise, where $X = 0$, $Y = 1 + i$ and $Z = -2$;

(b) C is the circle $|z - i| = 2$ oriented counterclockwise.

Solution. For part (a) we have

$$\begin{aligned}
\int_C \overline{z} dz &= \int_{XY} \overline{z} dz + \int_{YZ} \overline{z} dz + \int_{ZX} \overline{z} dz \\
&= \int_0^1 \underbrace{\overline{t(1+i)}}_{f(z(t))} \cdot \underbrace{(1+i)}_{z'(t)} \, dt \\
&\quad + \int_0^1 \underbrace{\overline{(1-t)(1+i) - 2t}}_{f(z(t))} \cdot \underbrace{(-3-i)}_{z'(t)} \, dt \\
&\quad + \int_0^1 \underbrace{\overline{-2(1-t)}}_{f(z(t))} \cdot \underbrace{2}_{z'(t)} \, dt \\
&= \int_0^1 2t \, dt + \int_0^1 ((2i - 4) + 10t) \, dt + \int_0^1 4(t-1) \, dt \\
&= 1 + (2i - 4) + 5 - 2 = 2i
\end{aligned}$$

For part (b) we have

$$\int_0^{2\pi} \underbrace{\overline{i + 2e^{it}}}_{f(z(t))} \cdot \underbrace{2ie^{it}}_{z'(t)} \, dt = \int_0^{2\pi} 2i(-i + 2e^{-it})e^{it} dt = 8\pi i.$$