SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 6

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(1) Using the appropriate definition of limits involving infinity, show that

(a) $\displaystyle\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4;$

(b) $\displaystyle\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty;$

(c) $\displaystyle\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty.$

Solution. (a) Write

$$\lim_{z\to 0} \frac{4\left(\frac{1}{z}\right)^2}{\left(\left(\frac{1}{z}\right)-1\right)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2} = 4.$$

(b) Here we write

$$\lim_{z\to 1} \frac{1}{1/(z-1)^3} = \lim_{z\to 1}(z-1)^3 = 0.$$

(c) Finally we write

$$\lim_{z\to 0} \frac{\frac{1}{z}-1}{\left(\frac{1}{z}\right)^2+1} = \lim_{z\to 0} = \frac{z-z^2}{1+z^2} = 0.$$

(2) Use the Wirtinger operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\big[\,(u_x - v_y) + i\,(v_x + u_y)\,\big] = 0$$

Thus derive the *complex form* $\partial f/\partial \overline{z} = 0$ *of the Cauchy-Riemann equations.*

Solution. Apply the Wirtinger operator to a function $f(z) = u(x, y) + iv(x, y)$. That is

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \\ &= \frac{1}{2}\,(u_x + iv_x) + \frac{i}{2}\,(u_y + iv_y) \\ &= \frac{1}{2}\big[\,(u_x - v_y) + i\,(v_x + u_y)\,\big]. \end{aligned}$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are satisfied, this tells us that $\partial f/\partial \overline{z} = 0$.

Note: we showed in class (Lecture 15, page 4) that $f'(z) = \partial f/\partial z$ for complex differentiable f. This should make intuitive sense; viewing f as a function of z and $\overline{z}$, from above we can see the Cauchy Riemann equations imply $\partial f/\partial \overline{z} = 0$.

(3) Determine which of the following functions $f(z)$ are entire and which are not? Justify your answer. If $f(z)$ is entire, find $f'(z)$.

(a) $f(z) = \dfrac{1}{1+|z|^2}$;

(b) $f(z) = (x^2 - y^2) + 2xyi$;

(c) $f(z) = (x^2 - y^2) - 2xyi$.

Solution. (a) Since $u(x,y) = (1 + x^2 + y^2)^{-1}$ and $v(x,y) = 0$,

$$u_x = -2x(1 + x^2 + y^2)^{-2}$$

which is only equal to v_y when $x = 0$. Hence the Cauchy-Riemann equations for $f(z)$ cannot hold in an entire neighbourhood, and thus it is not entire.

(b) Since

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = (2x + 2yi) + i(-2y + 2xi) = 0,$$

the Cauchy-Riemann equations hold for $f(z)$ everywhere. And since f_x and f_y are continuous, $f(z)$ is analytic on $\mathbb{C}$. And $f'(z) = 2x + 2yi = 2z$.

(c) Since

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = (2x - 2yi) + i(-2y - 2xi) = 4x - 4yi = 0,$$

only when $x, y = 0$. Hence the Cauchy-Riemann equations fail to hold in a whole neighbourhood, so $f(z)$ is not entire.

(4) Find the derivatives of the following functions in an appropriate domain:

(a) $f(z) = z \operatorname{Log} z$;

(b) $f(z) = \operatorname{Log}(z+1)$.

Solution. (a) From the differentiation rules we have that the function $z \operatorname{Log} z$ is differentiable at all points where both of the functions z and $\operatorname{Log} z$ are differentiable. Because z is entire and $\operatorname{Log} z$ is differentiable on the domain

$$|z| > 0, \ -\pi < \operatorname{Arg}(z) < \pi$$

it follows that $z \operatorname{Log} z$ is differentiable on the same domain.

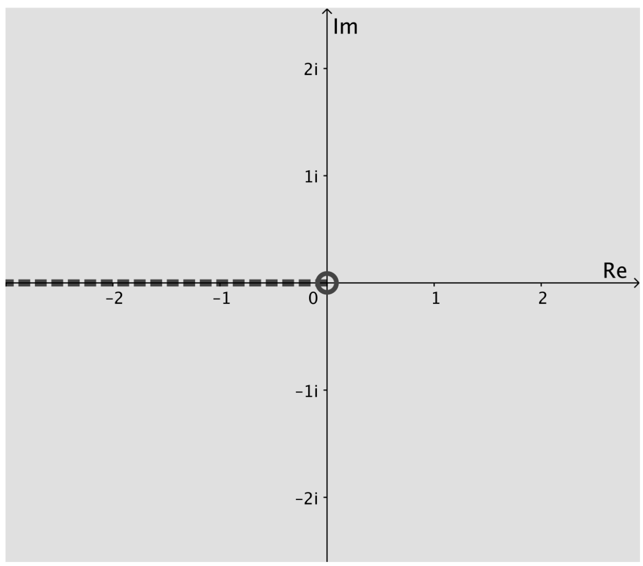

Figure 1: $z \operatorname{Log} z$ is not differentiable on the dashed ray.

In this domain the derivative is given by the product rule

$$\frac{d}{dz}(z \operatorname{Log} z) = z \cdot \frac{1}{z} + \operatorname{Log} z = 1 + \operatorname{Log} z$$

(b) The function $\text{Log}(z+1)$ is a composition of the functions $\text{Log}\, z$ and $z+1$. Because the function $z+1$ is entire, it follows from the chain rule that $\text{Log}(z+1)$ is differentiable at all points $w = z+1$ such that $|w| > 0,\ -\pi < \text{Arg}\,(w) < \pi$. In other words, this function is differentiable at the point w whenever w does not lie on the nonpositive real axis. To determine the corresponding values of z for which $\text{Log}(z+1)$ is not differentiable, we first solve for z in terms of w to obtain $z = w - 1$. The equation $z = w - 1$ defines a linear mapping of the w-plane onto the z-plane given by translation by -1. Under this mapping the nonpositive real axis is mapped onto the ray emanating from $z = -1$ and containing the point $z = -2$ shown in color in Figure **??**.

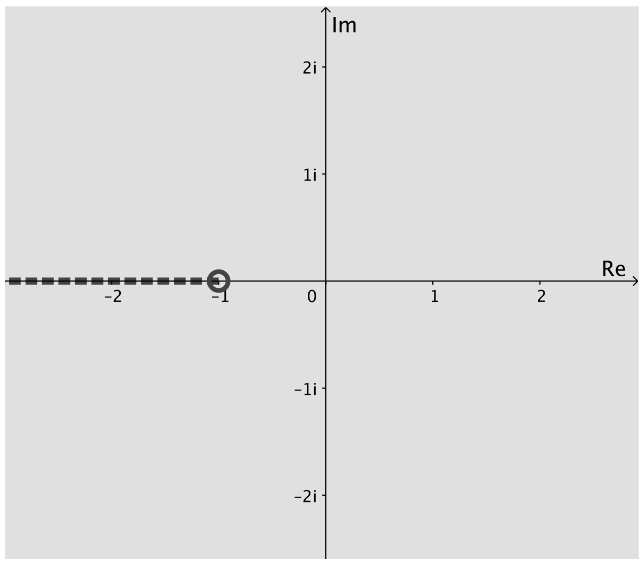

Figure 2: $\text{Log}(z+1)$ is not differentiable on the dashed ray.

Thus, if the point $w = z+1$ is on the nonpositive real axis, then the point z is on the dashed ray shown in Figure **??**. This implies that $\text{Log}(z+1)$ is differentiable at all points z that are not on this ray. For such points, the chain rule gives:

$$\frac{d}{dz}\text{Log}(z+1) = \frac{1}{z+1}.$$