SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 5

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(1) Show that the limit of the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

as z tends to 0 does not exist.

Hint: Do this letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin.

Solution. Consider the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2 = \left(\frac{x+iy}{x-iy}\right)^2 \qquad (z \neq 0),$$

where $z = x + iy$. Observe that if $z = (x, 0)$, then

$$f(z) = \left(\frac{x+i0}{x-i0}\right)^2 = 1$$

and if $z = (0, y)$

$$f(z) = \left(\frac{0+iy}{0-iy}\right)^2 = 1.$$

However, if $z = (x, x)$,

$$f(z) = \left(\frac{x+ix}{x-ix}\right)^2 = i^2 = -1.$$

This shows that $f(z)$ has value 1 at all nonzero points on the real and imaginary axes but value -1 at all nonzero points on the line $y = x$. Thus the limit of $f(z)$ as z tends to 0 cannot exist.

(2) Find $f'(z)$ when

(a) $f(z) = \dfrac{z-1}{2z+1}, \ (z \neq -1/2);$

(b) $f(z) = \dfrac{(1+z^2)^4}{z^2}, \ (z \neq 0).$

Solution. (a) If $f(z) = \dfrac{z-1}{2z+1}, \ (z \neq -1/2)$, then

$$\begin{aligned} f'(z) &= \frac{(2z+1)\frac{d}{dz}(z-1) - (z-1)\frac{d}{dz}(2z+1)}{(2z+1)^2} \\ &= \frac{3}{(2z+1)^2}. \end{aligned}$$

(b) If $f(z) = \dfrac{(1+z^2)^4}{z^2}, \ (z \neq 0)$, then

$$\begin{aligned} f'(z) &= \frac{z^2\frac{d}{dz}(1+z^2)^4 - (1+z^2)^4\frac{d}{dz}z^2}{(z^2)^2} \\ &= \frac{z^2 4(1+z^2)^3(2z) - (1+z^2)^4 2z}{z^4} \\ &= \frac{2(1+z^2)^3(3z^2-1)}{z^3}. \end{aligned}$$

(3) Determine where $f'(z)$ exists and find its value when

(a) $f(z) = \dfrac{1}{z}$;

(b) $f(z) = x^2 + iy^2$.

(c) $f(z) = z\,\mathrm{Im}(z)$.

Solution. (a) $f(z) = \dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$. Thus

$$u = \frac{x}{x^2+y^2} \text{ and } v = \frac{-y}{x^2+y^2}$$

are defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ (they are rational polynomial functions). So we have

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad u_y = \frac{-2xy}{(x^2+y^2)^2}$$

and

$$v_x = \frac{2xy}{(x^2+y^2)^2}, \quad v_y = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere except at $(0,0)$; they are also continuous and satisfy Cauchy-Riemann equations:

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2} = v_y \text{ and } u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x, \qquad (x^2+y^2 \neq 0).$$

Hence $f'(z)$ exists when $z \neq 0$. Moreover, when $z \neq 0$, we have

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} \\ &= -\frac{x^2 - i2xy - y^2}{(x^2+y^2)^2} = -\frac{(x-iy)^2}{(x^2+y^2)^2} \\ &= -\frac{(\overline{z})^2}{(z\overline{z})^2} = -\frac{(\overline{z})^2}{z^2(\overline{z})^2} \\ &= -\frac{1}{z^2}. \end{aligned}$$

(b) $f(z) = x^2 + iy^2$. Thus $u = x^2$ and $v = y^2$ are defined on $\mathbb{R}^2$ (they are polynomial functions). So we have

$$u_x = 2x, \quad u_y = 0$$

and

$$v_x = 0, \quad v_y = 2y.$$

The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere and they are also continuous.

Now considering Cauchy-Riemann equations

$$u_x = v_y \implies 2x = 2y \implies y = x$$

and

$$u_y = -v_x \implies 0 = 0,$$

we have that $f'(z)$ exists only when $y = x$, and we find that

$$f'(x + iy) = u_x(x, x) + iv_x(x, x) = 2x + i0 = 2x.$$

(c) $f(z) = z\,\mathrm{Im}(z) = (x + iy)y = xy + iy^2$. Here $u = xy$ and $v = y^2$, which are defined on $\mathbb{R}^2$ (they are polynomial functions). So we have

$$u_x = y, \quad u_y = x$$

and

$$v_x = 0, \quad v_y = 2y.$$

The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere and they are also continuous.

Now observe that

$$u_x = v_y \implies y = 2y \implies y = 0$$

and

$$u_y = -v_x \implies x = 0.$$

Hence $f'(z)$ exists only when $z = 0$. In fact

$$f'(0) = u_x(0, 0) + iv_x(0, 0) = 0 + i0 = 0.$$

(4) Show that each of these functions is differentiable in the indicated domain of definition, and also find $f'(z)$:

(a) $f(z) = \dfrac{1}{z^4}, \quad z \neq 0;$

(b) $f(z) = \sqrt{r}e^{i\theta/2}, \quad (r > 0, \alpha < \theta < \alpha + 2\pi).$

Solution. (a) $f(z) = \dfrac{1}{z^4} = \left(\dfrac{1}{r^4}\cos(4\theta)\right) + i\left(-\dfrac{1}{r^4}\sin(4\theta)\right)$, with $z \neq 0$. The first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere with $z \neq 0$; and they are also continuous.

Since

$$\text{C/R: } ru_r = -\frac{4}{r^4}\cos(4\theta) = v_\theta \text{ and } u_\theta = -\frac{4}{r^4}\sin(4\theta) = -rv_r,$$

f is analytic in its domain of definition. Furthermore,

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-\frac{4}{r^5}\cos(4\theta) + i\frac{4}{r^5}\sin(4\theta)\right) \\ &= -\frac{4}{r^5}e^{-i\theta}(\cos(4\theta) - i\sin(4\theta)) = -\frac{4}{r^5}e^{-i\theta}e^{-i4\theta} \\ &= \frac{-4}{r^5e^{i5\theta}} = -\frac{4}{(re^{i\theta})^5} = -\frac{4}{z^5}. \end{aligned}$$

(b) $f(z) = \sqrt{r}e^{i\theta/2} = \sqrt{r}\cos\dfrac{\theta}{2} + i\sqrt{r}\sin\dfrac{\theta}{2}, \quad (r > 0, \alpha < \theta < \alpha + 2\pi).$

The first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in its domain of definition; and they are also continuous.

Since

$$\text{C/R: } ru_r = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2} = v_\theta \text{ and } u_\theta = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} = -rv_r,$$

f is analytic in its domain of definition. Moreover,

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + i\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right) \\ &= \frac{1}{2\sqrt{r}}e^{-i\theta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2\sqrt{r}}e^{-i\theta}e^{i\theta/2} \\ &= \frac{1}{2\sqrt{r}e^{i\theta/2}} = \frac{1}{2f(z)}. \end{aligned}$$

(5) Show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$;

(b) $f(z) = 2xy + i(x^2 - y^2)$;

(c) $f(z) = e^y e^{ix}$.

Solution.

(a) Here $u = xy$ and $v = y$, which are defined on $\mathbb{R}^2$ (they are polynomial functions).

The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere and they are also continuous.

However, $f(z)$ is nowhere analytic since

$$u_x = v_y \implies y = 1 \text{ and } u_y = -v_x \implies x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0, 1) = i$.

(b) Here $u = 2xy$ and $v = x^2 - y^2$, which are defined on $\mathbb{R}^2$ (they are polynomial functions).

The first-order partial derivatives $u_x = 2y, u_y = 2x, v_x = 2x, v_y = -2y$ exist everywhere and they are also continuous. Observe

$$u_x = v_y \implies y = 0, \quad \text{and} \quad u_y = -v_x \implies x = 0$$

so the Cauchy Riemann equations hold only at $(0, 0)$, so f is nowhere analytic.

(c) $f(z) = e^y e^{ix} = e^y(\cos x + i \sin x) = e^y \cos x + ie^y \sin x$. Here $u = e^y \cos x$ and $v = e^y \sin x$, which are defined on $\mathbb{R}^2$ (they are trigonometric and exponential functions).

The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere; and they are also continuous.

However, $f(z)$ is nowhere analytic since

$$u_x = v_y \implies -e^y \sin x = e^y \sin x \implies 2e^y \sin x = 0 \implies \sin x = 0$$

and

$$u_y = -v_x \implies e^y \cos x = -e^y \cos x \implies 2e^y \cos x = 0 \implies \cos x = 0.$$

More precisely, the roots of the equation $\sin x = 0$ are $n\pi$ $(n \in \mathbb{Z})$, and $\cos(n\pi) = (-1)^n \neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.