SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 4

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(1) Find all values of z such that

(a) $e^z = -2$;

(b) $e^z = 1 + \sqrt{3}$.

Solution. (a) Write $e^z = -2$ as $e^x e^{iy} = 2e^{i\pi}$. This means that

$$e^x = 2 \quad \text{and} \quad y = \pi + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = (2n+1)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Therefore

$$z = \ln 2 + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Write $e^z = 1 + \sqrt{3}$ as $e^x e^{iy} = \left(1 + \sqrt{3}\right) e^{i \cdot 0}$, from which we deduce that

$$e^x = 1 + \sqrt{3} \quad \text{and} \quad y = 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln\left(1 + \sqrt{3}\right) \quad \text{and} \quad y = 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Consequently,

$$z = \ln\left(1 + \sqrt{3}\right) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(2) Show that $\overline{\exp(iz)} = \exp(i\overline{z})$ if and only if $z = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$.

Solution. We can solve this problem by finding all the roots of the equation

$$\overline{\exp(iz)} = \exp(i\overline{z}).$$

Set $z = x + iy$ and rewrite the equation as

$$e^{-y}e^{-ix} = e^{y}e^{ix}.$$

Now, recall that two nonzero complex numbers

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}$$

are equal if and only if

$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2n\pi,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Thus

$$e^{-y} = e^{y} \quad \text{and} \quad -x = x + 2n\pi,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Then

$$y = 0 \quad \text{and} \quad x = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The roots of the original equation are, therefore, $z = n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$.

(3) Show that

(a) $\text{Log}\ (1+i)^2 = 2\,\text{Log}\ (1+i)$;

(b) $\text{Log}\ (-1+i)^2 \neq 2\,\text{Log}\ (-1+i)$.

Solution. (a) Notice that

$$\text{Log}\ (1+i)^2 = \text{Log}\ (2i) = \ln 2 + \frac{\pi}{2}i$$

and

$$2\text{Log}\ (1+i) = 2\left(\ln\sqrt{2} + i\frac{\pi}{4}\right) = \ln 2 + \frac{\pi}{2}i.$$

Thus

$$\text{Log}\ (1+i)^2 = 2\,\text{Log}\ (1+i).$$

(b) For the second part, we have

$$\text{Log}\ (-1+i)^2 = \text{Log}\ (-2i) = \ln 2 - \frac{\pi}{2}i$$

and

$$2\,\text{Log}\ (-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + \frac{3\pi}{2}i.$$

Hence

$$\text{Log}\ (-1+i)^2 \neq 2\,\text{Log}\ (-1+i)$$

(4) Show that

(a) the set of values of $\log\left(i^{1/2}\right)$ is

$$\left(n+\frac{1}{4}\right)\pi i \qquad (n=0,\pm1,\pm2,\ldots);$$

(b) the set of values of $\log\left(i^2\right)$ is *not* the same as the set of values of $2\log i$.

Solution. (a) The two values of $i^{1/2}$ are $e^{i\pi/4}$ and $e^{i5\pi/4}$. Observe that

$$\log\left(e^{i\pi/4}\right)=\ln 1+i\left(\frac{\pi}{4}+2n\pi\right)=\left(2n+\frac{1}{4}\right)\pi i \qquad (n=0,\pm1,\pm2,\ldots)$$

and

$$\log\left(e^{i5\pi/4}\right)=\ln 1+i\left(\frac{5\pi}{4}+2n\pi\right)=\left[(2n+1)+\frac{1}{4}\right]\pi i \qquad (n=0,\pm1,\pm2,\ldots).$$

Combining these two sets of values, we obtain

$$\log\left(i^{1/2}\right)=\left(n+\frac{1}{4}\right)\pi i \qquad (n=0,\pm1,\pm2,\ldots).$$

Notice also that

$$\frac{1}{2}\log i=\frac{1}{2}\left[\ln 1+i\left(\frac{\pi}{2}+2n\pi\right)\right]=\left(n+\frac{1}{4}\right)\pi i \qquad (n=0,\pm1,\pm2,\ldots).$$

Thus the set of values of $\log\left(i^{1/2}\right)$ is the same as the set of values of $\frac{1}{2}\log i$, and thus we can write

$$\log\left(i^{1/2}\right)=\frac{1}{2}\log i.$$

(b) Notice that

$$\log\left(i^2\right)=\log(-1)=\ln 1+(\pi+2n\pi)i=(2n+1)\pi i \qquad (n=0,\pm1,\pm2,\ldots).$$

On the other hand we have

$$2\log i=2\left[\ln 1+i\left(\frac{\pi}{2}+2n\pi\right)\right]=(4n+1)\pi i \qquad (n=0,\pm1,\pm2,\ldots).$$

Therefore, the set of values of $\log\left(i^2\right)$ is not the same as the set of values of $2\log i$. In other words, $\log\left(i^2\right)\neq 2\log i$.

(5) Use the definition

$$z^c = \exp(c \log z) \qquad z \neq 0,$$

to show that $\left(-1+\sqrt{3}i\right)^{3/2} = \pm 2\sqrt{2}$.

Solution. Since $-1+\sqrt{3}i = 2e^{2\pi/3}$, we have that

$$\begin{aligned}\left(-1+\sqrt{3}i\right)^{3/2} &= \exp\left[\frac{3}{2}\log(-1+\sqrt{3}i)\right] = \exp\left\{\frac{3}{2}\left[\ln 2 + i\left(\frac{2\pi}{3}+2n\pi\right)\right]\right\} \\ &= \exp\left[\ln\left(2^{3/2}\right)+(3n+1)\pi i\right] = 2\sqrt{2}\exp\left[(3n+1)\pi i\right],\end{aligned}$$

where $n = 0, \pm1, \pm2, \ldots$.

Now, observe that if n is even, then $3n+1$ is odd; and so $\exp[(3n+1)\pi i] = -1$. On the other hand, if n is odd, $3n+1$ is even; and this means that $\exp[(3n+1)\pi i] = 1$. So we obtained only two distinct values of $\left(-1+\sqrt{3}i\right)^{3/2}$. Specifically,

$$\left(-1+\sqrt{3}i\right)^{3/2} = \pm 2\sqrt{2}.$$