SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 3

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(1) Find the principal argument Arg z when

$$\text{(a) } z = \frac{i}{-2-2i}; \qquad \text{(b) } z = \left(\sqrt{3}-i\right)^6.$$

Solution. (a) Since

$$\arg\left(\frac{i}{-2-2i}\right) = \arg i - \arg\ (-2-2i)\,,$$

one value of $\arg\left(\frac{i}{-2-2i}\right)$ is

$$\frac{\pi}{2} - \left(-\frac{3\pi}{4}\right), \quad \text{or} \quad \frac{5\pi}{4}.$$

Hence the principal value is

$$\frac{5\pi}{4} - 2\pi, \quad \text{or} \quad -\frac{3\pi}{4}.$$

(b) Now, since

$$\arg\left(\sqrt{3}-i\right)^6 = 6\arg\left(\sqrt{3}-i\right)$$

one value of $\arg\left(\sqrt{3}-i\right)^6$ is

$$6\left(-\frac{\pi}{6}\right); \quad \text{or} \quad -\pi.$$

So the principal value is $-\pi + 2\pi = \pi$.

(2) In each case, find all the roots in rectangular coordinates:

$$\text{(a)}\ \ (-16)^{1/4}; \qquad \text{(b)}\ \ (-1)^{1/3}.$$

Solution. (a) Since $-16 = 16\exp\left[i\left(\pi + 2k\pi\right)\right]$ with $k = 0, \pm 1, \pm 2, \ldots$, the roots are

$$(-16)^{1/4} = 2\exp\left[i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right] \qquad k = 0, 1, 2, 3.$$

Thus, for $k = 0$ we have the first root

$$c_0 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}(1+i)$$

This is know as the *principal root.* The other three roots are

$$c_1 = -\sqrt{2}(1-i)$$

$$c_2 = -\sqrt{2}(1+i)$$

$$c_3 = \sqrt{2}(1-i).$$

(b) By writing $-1 = 1\exp\left[i\left(\pi + 2k\pi\right)\right]$ with $k = 0, \pm 1, \pm 2, \ldots$, we see that

$$(-1)^{1/3} = \exp\left[i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right] \qquad k = 0, 1, 2.$$

The principal root is

$$c_0 = e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1+\sqrt{3}i}{2}$$

The other roots are

$$c_1 = e^{i\pi} = -1$$

$$c_2 = e^{i5\pi/3} = e^{i2\pi}e^{-i\pi/3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1-\sqrt{3}i}{2}.$$

(3) Find the Möbius transformation that maps the points

$$z_1 = \infty, \quad z_2 = i, \quad z_3 = 0$$

onto the points

$$w_1 = 0, \quad w_2 = i, \quad w_3 = \infty.$$

Solution. We need to find the values a, b, c and d in

$$T(z) = \frac{az+b}{cz+d}, \text{ with } ad - bc \neq 0.$$

Since $T(\infty) = 0$, then

$$\frac{a}{c} = 0 \implies a = 0 \quad (c \neq 0).$$

Now, since $T(0) = \infty$, then

$$\frac{-d}{c} = 0 \implies d = 0 \quad (c \neq 0)$$

Finally, since $T(i) = i$, we have

$$\frac{ai+b}{ci+d} = i.$$

Using previous values, we obtain

$$\frac{b}{ci} = i \implies b = i^2 c = -c.$$

Hence, the general Möbius transformation is

$$T(z) = \frac{-c}{cz}$$

In particular, for $c = 1$ we have

$$T(z) = -\frac{1}{z}.$$

(4) Find the Möbius transformation that maps the points

$$z_1 = -1, \quad z_2 = \infty, \quad z_3 = i$$

onto the points

$$w_1 = \infty, \quad w_2 = i, \quad w_3 = 1.$$

Solution. Since $T(\infty) = i$, then

$$\frac{a}{c} = i \implies a = ic \implies ai = -c \quad (c \neq 0).$$

Now, since $T(-1) = \infty$, then

$$\frac{-d}{c} = -1 \implies d = c \quad (c \neq 0).$$

Finally, since $T(i) = 1$, we have

$$\frac{ai + b}{ci + d} = 1.$$

Using previous values, we obtain

$$\frac{-c + b}{ci + c} = 1 \implies b - c = c(i + 1) \implies b = c(i + 2).$$

Hence, the general Möbius transformation is

$$T(z) = \frac{icz + c(i + 2)}{cz + c}.$$

In particular, for $c = 1$ we have

$$T(z) = \frac{iz + (i + 2)}{z + 1}.$$