SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401/3901
Tutorial Worksheet
Semester 1, 2024, Week 2

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(1) Prove that multiplication of complex numbers is commutative.

Solution. Consider $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ complex numbers. Now

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) \\ &= (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2) \\ &= (x_2 x_1 - y_2 y_1, y_2 x_1 + x_2 y_1) \\ &= (x_2, y_2)(x_1, y_1) \\ &= z_2 z_1. \end{aligned}$$

This proves that multiplication of complex numbers is commutative.

(2) Simplify each of these to a real number:

$$\text{(a) } \frac{1+2i}{3-4i} + \frac{2-i}{5i}; \qquad \text{(b) } \frac{5i}{(1-i)(2-i)(3-i)}.$$

Solution.

$$\begin{aligned} \text{(a) } \frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)(-5i)}{(5i)(-5i)} \\ &= \frac{-5+10i}{25} + \frac{-5-10i}{25} = -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} \text{(b) } \frac{5i}{(1-i)(2-i)(3-i)} &= \frac{5i}{(1-3i)(3-i)} \\ &= \frac{5i}{-10i} = -\frac{1}{2} \end{aligned}$$

(3) Use the properties of conjugates and moduli to show that

$$\text{(a) } \overline{\overline{z}+3i} = z - 3i; \quad \text{(b) } \overline{iz} = -i\,\overline{z}; \quad \text{(c) } \left|(2\overline{z}+5)\left(\sqrt{2}-i\right)\right| = \sqrt{3}|2z+5|.$$

Solution.

$$\begin{aligned}\text{(a) } \overline{\overline{z}+3i} &= \overline{\overline{z}} + \overline{3i} \\ &= z - 3i.\end{aligned}$$

$$\text{(b) } \overline{iz} = \overline{i}\,\overline{z} = -i\overline{z}.$$

$$\begin{aligned}\text{(c) } \left|(2\overline{z}+5)\left(\sqrt{2}-i\right)\right| &= |2\overline{z}+5|\left|\sqrt{2}-i\right| \\ &= \left|\overline{2z+5}\right|\sqrt{2+1} \\ &= \sqrt{3}|2z+5|.\end{aligned}$$

(4) Verify that $\sqrt{2}|z| \geq |\text{Re } z| + |\text{Im } z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Solution. To verify the inequality $\sqrt{2}|z| \geq |\text{Re } z| + |\text{Im } z|$, write $z = x + iy$, with $x = \text{Re } z$ and $y = \text{Im } z$. Hence, we need to show

$$\begin{aligned}\sqrt{2}\sqrt{x^2+y^2} &\geq |x| + |y|, \text{or on squaring both sides} \\ 2(x^2+y^2) &\geq |x|^2 + 2|x||y| + |y|^2.\end{aligned}$$

Since all quantities are non-negative, these expressions are equivalent. On subtracting the RHS from both sides, this becomes equivalent to

$$\begin{aligned}|x|^2 - 2|x||y| + |y|^2 &\geq 0, \text{i.e.,} \\ (|x| - |y|)^2 &\geq 0.\end{aligned}$$

This last inequality is true (square of a real number), hence so is the original inequality.

(5) Use de Moivre's formula to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; (b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

Solution. We know from de Moivre's formula that

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

or

$$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

In other words,

$$(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta$$

By equating the corresponding real and imaginary parts, we arrive at the desire trigonometric identities.