SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 10

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(1) Determine where the function $f(z) = z - e^{-z} + 1 - i$ is conformal.

(2) Find all points where the function $f(z) = \sin z$ is conformal.

(3) Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic at a point $z_0 \in \mathbb{C}$. The *angle of rotation* of f at z_0 is defined to be the argument of $f'(z_0)$.

We motivate this definition as follows: take a curve C passing through z_0. Let $z(t)$ be a parametrisation of the curve in a neighbourhood of z_0; namely

- *$z(t)$ on $[a, b]$ traces a portion of C, and*
- *some $t_0 \in (a, b)$ returns $z(t_0) = z_0$.*

By the chain rule $\frac{d}{dt}\big(f(z(t))\big) = f'(z(t))z'(t)$. Thus

$$\arg[f'(z(t_0))] = \arg[f'(z(t_0))] + \arg[z'(t_0)].$$

The LHS is a tangent vector of the curve $f(C)$ at $f(z_0)$ and $z'(t_0)$ is a tangent vector of C at z_0. Thus $\arg[f'(z(t_0))]$ measures the change of argument. Observe this quantity is independent of the curve C we chose to analyse. See Section 112 in the Brown and Churchill textbook for further discussion).

Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n$ $(n = 1, 2, \ldots)$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point.

(4) Find a function harmonic in the upper half of the z-plane, $\mathrm{Im}\, z > 0$, which takes the values on the x axis: $G(x) = 1$ for $x > 0$, and $G(x) = 0$ for $x < 0$.

(5) Find a function harmonic inside the unit circle $|z| = 1$ and taking the values $F(\theta) = 1$ for $0 < \theta < \pi$, and $F(\theta) = 0$ for $\pi < \theta < 2\pi$ on its circumference.