SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Tutorial Worksheet
Semester 1, 2024, Week 10

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(1) Determine where the function $f(z) = z - e^{-z} + 1 - i$ is conformal.

Solution: It is easy to verify that $f(z) = z - e^{-z} + 1 - i$ is entire on $\mathbb{C}$. Now $f'(z) = 1 + e^{-z} = 0$ if

$$\begin{aligned} e^z(1 + e^{-z}) &= e^z(0) \\ e^z + 1 &= 0 \\ e^z &= -1 \\ z &= \log(-1) = \ln|-1| + i\arg(-1) \\ z &= i(\pi + 2\pi n) \\ z &= (2n+1)\pi i, \end{aligned}$$

for $n \in \mathbb{Z}$. Hence, f is conformal for all z except $z = (2n+1)\pi i$, $n \in \mathbb{Z}$.

(2) Find all points where the function $f(z) = \sin z$ is conformal.

Solution: The function $f(z) = \sin z$ is entire on $\mathbb{C}$ and we have that $f'(z) = \cos z$. Now $\cos z = 0$ if and only if $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$ and so each of these points is a critical point of f. Therefore, f is conformal for all z except $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$.

(3) Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n$ $(n = 1, 2, \ldots)$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point.

Solution: Notice that $f'(z) = nz^{n-1}$. Now for $z_0 = r_0 e^{i\theta_0}$ we have

$$f'\left(r_0 e^{i\theta_0}\right) = n'\left(r_0 e^{i\theta_0}\right)^{n-1}.$$

Thus the angle of rotation is

$$\arg\left[f'\left(r_0 e^{i\theta_0}\right)\right] = \arg\left[n\left(r_0 e^{i\theta_0}\right)^{n-1}\right] = (n-1)\theta_0$$

and the scale factor is

$$\left|f'\left(r_0 e^{i\theta_0}\right)\right| = \left|n r_0^{n-1} e^{i(n-1)\theta_0}\right| = n r_0^{n-1}.$$

See Section 112 in the Brown and Churchill textbook for further discussion on the angle of rotation.

(4) Find a function harmonic in the upper half of the z-plane, Im $z > 0$, which takes the prescribe values on the x axis given by $G(x) = 1$ for $x > 0$, and $G(x) = 0$ for $x < 0$.

Solution: We need to find $\Phi(x, y)$ such that

$$\nabla^2\Phi = 0, \quad -\infty < x < \infty, \ y > 0.$$
$$\Phi = 1, \ x > 0; \quad \Phi = 0, \ x < 0.$$

This is a Dirichlet problem for the upper half plane.

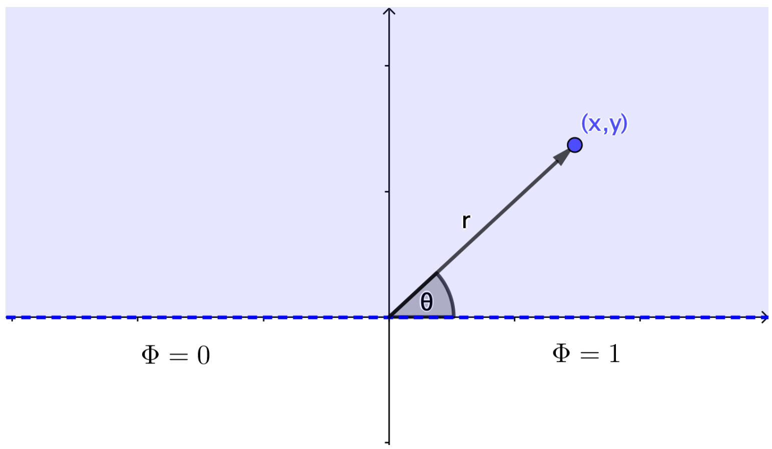

Figure 1: Upper half plane.

The function $A\theta + B$, where A and B are real constants, is harmonic since it is the imaginary part of $A\log(z)+Bi$. To determine A and B note that the boundary conditions are $\Phi = 1$ for $x > 0$, that is, $\theta = 0$ and $\Phi = 0$ for $x < 0$, that is $\theta = \pi$. Thus

$$1 = A(0) + B, \qquad 0 = A(\pi) + B$$

from which $A = -1/\pi$, $B = 1$. Then the required solution is

$$\Phi = A\theta + B = 1 - \frac{\theta}{\pi} = 1 - \frac{1}{\pi}\text{Arg } z$$

We can write this as

$$\Phi(x, y) = 1 - \frac{1}{\pi}\arctan(y/x)$$

for $y > 0$, where we take arctan to be the inverse of tan taking values in $(0, \pi]$. Notice for this arctan,

$$\lim_{x\to 0^+} \arctan(x) = 0, \qquad \lim_{x\to 0^-} \arctan(x) = \pi$$

(indeed, draw a graph of $y = \tan(x)$ with x restricted to $(0, \pi]$). It follows that

$$\lim_{y\to 0^+} \Phi(x, y) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

(5) Find a function harmonic inside the unit circle $|z| = 1$ and taking the prescribed values given by $F(\theta) = 1$ for $0 < \theta < \pi$, and $F(\theta) = 0$ for $\pi < \theta < 2\pi$ on its circumference.

Solution: This is a Dirichlet problem for the unit circle in which we need to find a function satisfying Laplace's equation inside $|z| = 1$ and taking the values 1 on the upper arc of the circle and 0 on the lower arc of the circle.

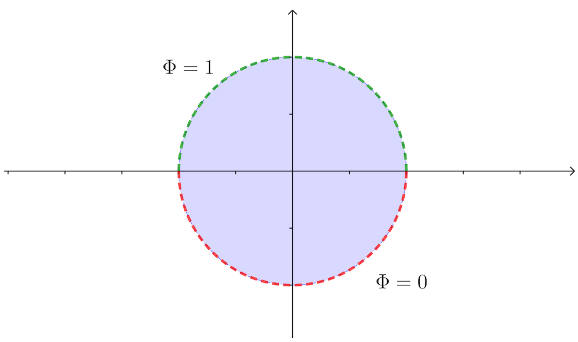

Figure 2: Unit circle.

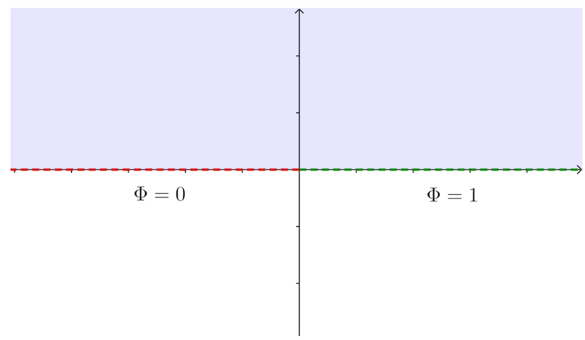

Figure 3: Image under $w = i(1-z)/(1+z)$.

We map the interior of the circle $|z| = 1$ on to the upper half of the w plane by using the mapping

$$w = i\frac{1-z}{1+z}.$$

Under this transformation, the upper and lower arcs are mapped on to the positive and negative real axis on the w-plane respectively. This means that the boundary conditions $\Phi = 1$ on the upper arc of the circle and $\Phi = 0$ on the lower arc of the circle become respectively $\Phi = 1$ for $u > 0$ and $\Phi = 0$ for $u < 0$.

Thus we have reduced the problem to finding a function Φ harmonic in the upper half w-plane and taking the values 0 for $u < 0$ and 1 for $u > 0$. But this problem has already been solved in the previous exercise, and the solution (replacing x by u and y by v) is given by

$$\Phi = 1 - \frac{1}{\pi}\arctan\left(\frac{v}{u}\right). \qquad (1)$$

Now from $w = i(1-z)/(z+1)$, we find

$$u = \frac{2y}{(1+x)^2 + y^2}, \quad v = \frac{1-(x^2+y^2)}{(1+x)^2 + y^2}.$$

Then substituting these in (**??**), we find the required solution

$$\Phi = 1 - \frac{1}{\pi} \arctan \left(\frac{1 - (x^2 + y^2)}{2y} \right)$$

or in polar coordinates (r, θ)

$$\Phi = 1 - \frac{1}{\pi} \arctan \left(\frac{1 - r^2}{2r \sin \theta} \right).$$